

# LIE SUPERALGEBRA MODULES OF CONSTANT JORDAN TYPE

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**ABSTRACT.** The theories of  $\pi$ -points and modules of constant Jordan type have been a topic of much recent interest in the field of finite group scheme representation theory. These theories allow for a finite group scheme module  $M$  to be restricted down and considered as a module over a space of small subgroups whose representation theory is completely understood, but still provide powerful global information about the original representation of  $M$ .

This paper provides an extension of these ideas and techniques to study finite dimensional supermodules over a classical Lie superalgebra  $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ . Definitions and examples of  $\mathfrak{g}$ -modules of constant super Jordan type are given along with proofs of some properties of these modules. Additionally, endotrivial modules (a specific case of modules of constant Jordan type) are studied. The case when  $\mathfrak{g}$  is a detecting subalgebra, denoted  $\mathfrak{f}_r$ , of a stable Lie superalgebra is considered in detail and used to construct super vector bundles over projective space  $\mathbb{P}^{r-1}$ . Finally, a complete classification of supermodules of constant super Jordan type are given for  $\mathfrak{f}_1 = \mathfrak{sl}(1|1)$ .

## 1. INTRODUCTION

The study of modules of constant Jordan type was initiated in [8] by Carlson, Friedlander, and Pevtsova for representations of finite group schemes over a field of characteristic  $p > 0$ . Leading up to the study of these modules Friedlander and Pevtsova had previously developed a theory of  $p$ -points and the more generalized  $\pi$ -points in [19] and [20] where they study certain maps  $K[t]/t^p \rightarrow KG$  into the group algebra  $KG$  of a finite groups scheme  $G$ . The representations of  $K[t]/t^p$  can be completely classified by the so called Jordan type of the matrix representing a  $K[t]/t^p$ -module.

The simplicity of the  $K[t]/t^p$ -modules is used quite successfully to study  $G$  by restricting  $KG$ -modules along these maps and, under an equivalence relation, these maps are the so called  $\pi$ -points. The set of  $\pi$ -points,  $\Pi(G)$ , admits a scheme structure which is proven to be isomorphic to  $\text{Proj } H^\bullet(G, k)$  and the  $\pi$ -points detect projectivity in the sense that a  $KG$ -module  $M$  is projective if and only if it is projective when restricted to each  $\pi$ -point.

An interesting result of  $\pi$ -point theory was the introduction of modules of constant Jordan type. These modules are ones where the Jordan type (i.e. the isomorphism class of a nilpotent matrix representing  $t$  in the map  $K[t]/t^p \rightarrow KG$ ) is constant over all  $\pi$ -points in  $\Pi(G)$ . Some of the supporting theory is developed in [22] and in [8] it is shown that these modules are closed under the operations of direct sums, direct summands, tensor products, dualizing, and the syzygy operation of Heller shifts. This class of module also happens to contain the class of endotrivial modules which arise as modules of a particular Jordan type, and in a similar manner to projectivity detection, a  $kG$ -module is endotrivial if and only if it is endotrivial when restricted to each  $\pi$ -point.

These modules have further gone on to be studied and applied in [1], [3], [9], and [21] as well as numerous others. Interesting problems and applications related to these modules are constructing non-trivial examples, determining that certain Jordan types cannot exist, and

constructing (low rank) vector bundles through functors from finitely generated modules to quasi-coherent sheaves.

The main goal of this paper is to extend the theory and results on modules of constant Jordan type to the field of Lie superalgebra representation theory. Let  $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$  be a classical Lie superalgebra over an algebraically closed field  $k$  of characteristic 0. The category of finite dimensional supermodules which are completely reducible over  $\mathfrak{g}_{\bar{0}}$ , denoted  $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ , has been widely studied in [4], [5], [6], [7], [18], [26], [29], and is the category of interest in the author's previous work on endotrivial supermodules in [31] and [32].

There is no general analogue for  $\pi$ -point theory for an arbitrary Lie superalgebra, so an alternative method must be found or developed first. The approach taken in this paper is to introduce the notion of a supermodule of constant Jordan type using the self commuting cone

$$\mathcal{X} = \{x \in \mathfrak{g}_{\bar{1}} \mid [x, x] = 0\}$$

of a Lie superalgebra  $\mathfrak{g}$  considered by Duflo and Serganova in [18] which allows for a natural definition as follows. For all  $x \in \mathcal{X} \setminus \{0\}$ ,  $U(\langle x \rangle) \cong k[t]/t^2$  and so restricting a supermodule  $M \in \mathcal{F}$  to  $\langle x \rangle$  allows us to associate to each point  $x \in \mathcal{X}$  a Jordan type of  $M|_{\langle x \rangle}$ , thus giving a natural definition of supermodules of constant Jordan type. This situation is similar to the study of  $KG$ -modules of constant Jordan type over a field  $K$  of characteristic 2. One notable difference in this setting is that although the Jordan decomposition at a point  $x \in \mathcal{X}$  has blocks of size either 1 or 2, corresponding to trivial and projective  $U(\langle x \rangle)$ -supermodules respectively, the trivial modules can be concentrated in either even or odd degree leading to the notion of a super Jordan type.

This definition allows for many results to be proved about  $\mathfrak{g}$ -supermodules of constant Jordan type, some analogous to the finite group schemes setting and some specific to Lie superalgebras. Section 2 gives background information and presents some of the theory which is used in the remainder of the paper. Section 3 defines, introduces first properties, and gives examples of modules of constant Jordan type in  $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ . These modules are shown to be closed under direct sums, duals, tensor products, homomorphisms of supermodules, and the syzygy operation, but strikingly not closed under taking direct summands (an issue that is further considered and rectified in Section 5). Additionally there are results strictly relating to Lie superalgebras given. Proposition 3.5 shows that modules of constant Jordan type must lie in a block of maximal atypicality when  $\mathfrak{g}$  is a simple basic classical Lie superalgebra and Theorem 3.8 shows that, in many cases, the condition for a module to be of constant Jordan type can be checked using a finite number of points.

The topic of endotrivial modules is considered in Section 4, as again in this setting these are a specific case of modules of constant Jordan type. Theorem 4.1 shows that a module  $M$  is endotrivial if and only if it is endotrivial when restricted to each point  $x \in \mathcal{X}$ , i.e. a module  $M$  being endotrivial is equivalent to having constant Jordan type with exactly one trivial summand (and any number of projective summands).

Section 5 deals with modules of constant Jordan type over

$$\mathfrak{f} = \mathfrak{f}_r := \mathfrak{sl}(1|1) \times \dots \times \mathfrak{sl}(1|1)$$

where there are  $r$  copies of  $\mathfrak{sl}(1|1)$ . The Lie superalgebra  $\mathfrak{f}$  is the detecting subalgebra of a stable Lie superalgebra  $\mathfrak{g}$  as introduced in [5]. These subalgebras are analogues of elementary abelian subgroups in finite group representation theory in that they detect the cohomology of

$\mathfrak{g}$  in the sense that the cohomology ring of  $\mathfrak{g}$  embeds into a specific subring of the cohomology of  $\mathfrak{f}$ .

When working in  $\mathcal{F}_{(\mathfrak{f}, \mathfrak{f}_{\overline{0}})}$ , it is natural to expand the variety  $\mathcal{X}$  to include all of  $\mathfrak{f}_{\overline{1}}$  and in doing so allows a proof of closure of modules of constant Jordan type under direct summands in this case. Another benefit of this adjustment is that it allows for a generalization of the construction of vector bundles over projective space given in [21]. The vector bundles constructed from modules of constant super Jordan type in Theorem 5.5 inherit a natural  $\mathbb{Z}_2$  grading from the module structure and are called algebraic super vector bundles over projective space. These differ from super vector bundles over a super manifold (whose construction is explicitly shown to fail in Section 5.2.1) in that the base space is ungraded and is subsequently treated as a purely even object.

The last major result (Theorem 5.9) gives a classification of all modules of constant Jordan type for the specific case of  $\mathfrak{f}_1 = \mathfrak{sl}(1|1)$ . The theorem shows that the only  $\mathfrak{f}_1$ -modules of constant Jordan type are in fact the endotrivial modules which are isomorphic to the  $W$  modules of height 2 and their duals as defined in [9]. The proof is completed by exploiting the grading of a  $\mathfrak{f}_1$ -module and using the theory of generic kernels and images presented in [9]. This result also recovers a significant step ([31, Theorem 5.12]) in the proof of one of the main results in [31].

## 2. NOTATION AND PRELIMINARIES

Let  $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$  be a finite dimensional classical Lie superalgebra over an algebraically closed field  $k$  of characteristic 0. Here, classical means there is a connected reductive algebraic group  $G_{\overline{0}}$  such that  $\text{Lie}(G_{\overline{0}}) = \mathfrak{g}_{\overline{0}}$ , and an action of  $G_{\overline{0}}$  on  $\mathfrak{g}_{\overline{1}}$  that differentiates to the adjoint action of  $\mathfrak{g}_{\overline{0}}$  on  $\mathfrak{g}_{\overline{1}}$ . Note, if  $\mathfrak{g}$  is classical, then  $\mathfrak{g}_{\overline{0}}$  is reductive as a Lie algebra and  $\mathfrak{g}_{\overline{1}}$  is semisimple as a  $\mathfrak{g}_{\overline{0}}$ -module, but it is not assumed that  $\mathfrak{g}$  is simple. There are a number of module categories associated to  $\mathfrak{g}$  but one category, denoted  $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$ , is of particular interest because of its desirable properties and is defined as follows.

Let  $\mathfrak{t} \subseteq \mathfrak{g}$  be Lie superalgebras. Then define  $\mathcal{F}_{(\mathfrak{g}, \mathfrak{t})}$  to be the full subcategory of finite dimensional  $\mathfrak{g}$ -supermodules which are completely reducible over  $\mathfrak{t}$ . This category has enough projectives and injectives, and is self injective. Of primary concern is  $\mathcal{F} := \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$ , as it has been widely studied, and a number of the results on this category are used here. In the case that  $\mathfrak{g}_{\overline{0}}$  is a semisimple Lie algebra, then this is simply the category of all finite dimensional  $\mathfrak{g}$ -supermodules. By convention, supermodules are the only objects considered here and thus may be referred to as modules from now on.

**2.1. Atypicality and Blocks.** Much of the theory in what follows does not rely on the Lie superalgebra  $\mathfrak{g}$  having a non-degenerate bilinear form, i.e. is basic classical, but when one does exist (in all cases except for the Type I superalgebra  $P(n)$ ), then we may say a bit more about modules of constant Jordan type by making observations about certain combinatorial invariants.

Let  $\mathfrak{g}$  be a basic classical Lie superalgebra whose non-degenerate bilinear form is denoted as  $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow k$ . By fixing a Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}_{\overline{0}}$ ,  $\mathfrak{g}$  can be decomposed into root spaces, indexed by  $\Phi$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$$

each of which are homogeneous and one dimensional. Thus, we can define a parity associated to each root by assigning it to be the parity of the corresponding root space, and denote this decomposition by writing  $\Phi = \Phi_{\bar{0}} \sqcup \Phi_{\bar{1}}$ . We also fix a choice of a Borel subalgebra  $\mathfrak{b}$  containing  $\mathfrak{h}$  which defines positive and negative roots. Then define

$$\rho = \frac{1}{2} \sum_{\alpha_{\bar{0}} \in \Phi_{\bar{0}}^+} \alpha_{\bar{0}} - \frac{1}{2} \sum_{\alpha_{\bar{1}} \in \Phi_{\bar{1}}^+} \alpha_{\bar{1}}$$

where  $\Phi_i^+$  denotes the positive roots in  $\Phi_i$  relative to the choice of  $\mathfrak{b}$ .

The form  $(\cdot, \cdot)$  induces a form on  $\mathfrak{h}^*$  (denoted in the same way) and thus on the roots as well. We say that a root  $\alpha$  (necessarily odd) is *isotropic* if  $(\alpha, \alpha) = 0$ . Then the maximal number of isotropic, pairwise orthogonal roots is called the *defect of  $\mathfrak{g}$*  denoted  $\text{def}(\mathfrak{g})$  and does not depend on the choice of  $\mathfrak{h}$ , hence is well defined. Additionally, for a weight  $\lambda \in \mathfrak{h}^*$ , the maximal number of isotropic, pairwise orthogonal roots  $\alpha$  such that  $(\lambda + \rho, \alpha) = 0$  is called the *atypicality of  $\lambda$*  and denoted as  $\text{atyp}(\lambda)$ . This number again does not depend on any choices made and so is also well defined. By construction,  $\text{atyp}(\lambda) \leq \text{def}(\mathfrak{g})$  for any weight  $\lambda$ .

When  $\mathfrak{g}$  is also Type I, then  $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$  is a highest weight category and the simple modules are parameterized by a set  $X^+ \subseteq \mathfrak{h}^*$  and so any simple module is a highest weight module and is denoted  $L(\lambda)$  where  $\lambda \in X^+$ . In this case we define the *atypicality of  $L(\lambda)$*  to be the atypicality of the weight  $\lambda$  and write  $\text{atyp}(L(\lambda))$ .

If  $Z$  denotes the center of the universal enveloping algebra  $U(\mathfrak{g})$ , then for any weight  $\lambda$  we can define an algebra homomorphism  $\chi_\lambda : Z \rightarrow k$  which defines a central character. These characters decompose the category  $\mathcal{F}$  into blocks

$$\mathcal{F} = \bigoplus \mathcal{F}^\chi$$

in the sense that any indecomposable module  $M$  is contained in exactly one  $\mathcal{F}^\chi$  and we say that  $M$  admits the character  $\chi$ . The atypicality of a central character  $\chi$  is the maximal atypicality of a weight  $\lambda$  such that  $\chi_\lambda = \chi$ . One important property of these blocks is that [23] gives an equivalence of blocks of finite dimensional  $\mathfrak{g}$ -modules which admit certain characters.

**Theorem 2.1** (Gruson-Serganova). *Let  $\lambda$  be a dominant weight with atypicality  $\ell$ , then the block  $\mathcal{F}^{\chi_\lambda}$  is equivalent to the maximal atypical block of  $\mathfrak{g}_\ell$  containing the trivial module, where*

- if  $\mathfrak{g} = \mathfrak{gl}(m|n)$  then  $\mathfrak{g}_\ell = \mathfrak{gl}(\ell|\ell)$ ;
- if  $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$  then  $\mathfrak{g}_\ell = \mathfrak{osp}(2\ell+1|2\ell)$ ;
- if  $\mathfrak{g} = \mathfrak{osp}(2m|2n)$  then  $\mathfrak{g}_\ell = \mathfrak{osp}(2\ell|2\ell)$  or  $(2\ell+2|2\ell)$ .

Thus up to some equivalence, modules of a fixed atypicality (see Proposition 3.5) lie in the principal block of a fixed Lie superalgebra.

**2.2. The Associated Variety.** The representation theory of the one dimensional abelian Lie superalgebra generated by one odd element is completely understood. As verified by direct calculation in [5, Section 5.2], if  $\mathfrak{g} = \mathfrak{g}_{\bar{1}} = \langle x \rangle$  is a one dimensional Lie superalgebra, then there are only four indecomposable non-isomorphic modules (or two if the parity change is ignored). If  $\Pi$  denotes the parity change functor and  $k_{ev}$  the trivial module concentrated degree  $\bar{0}$  and  $k_{od} = \Pi(k_{ev})$ , then the four indecomposable  $U(\langle x \rangle)$  modules are  $k_{ev}$  and  $k_{od}$ ,

and their two dimensional projective covers which are isomorphic to  $U(\langle x \rangle)$  and  $\Pi(U(\langle x \rangle))$  respectively.

Dufflo and Sergonova define the variety

$$\mathcal{X} = \{x \in \mathfrak{g}_{\overline{1}} \mid [x, x] = 0\}$$

in [18]. If we view  $\mathfrak{g}_{\overline{1}}$  as an affine variety endowed with the Zariski topology, then  $\mathfrak{g}_{\overline{1}} \cong \mathbb{A}^{\dim \mathfrak{g}_{\overline{1}}}$ , and  $\mathcal{X}$  is a Zariski closed  $G_{\overline{0}}$ -invariant cone in  $\mathfrak{g}_{\overline{1}}$  and is referred to as the *self commuting cone*. The condition that  $[x, x] = 0$  is equivalent to the condition that  $x^2 = 0$  in the universal enveloping algebra  $U(\mathfrak{g})$  and so restricting a module  $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$  to an element  $x \in \mathcal{X}$  is reminiscent of the notion of Friendlander and Pevtsova's  $\pi$ -points (see [20]).

For  $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$ , the authors of [18] further define a subvariety of  $\mathcal{X}$  by viewing multiplication by  $x \in \mathcal{X}$  as a linear map from  $M$  to itself and define  $M_x = \text{Ker}(x) / \text{Im}(x)$  which is called the fiber of  $M$  at  $x$  and set

$$\mathcal{X}_M = \{x \in \mathcal{X} \mid M_x \neq 0\}.$$

By the characterization of the representations of  $\langle x \rangle$  for  $x \in \mathcal{X}$  given above, it can be seen that the previous definition is equivalent to the following

$$(2.2) \quad \mathcal{X}_M = \{x \in \mathcal{X} \mid M|_{\langle x \rangle} \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}.$$

The so called *associated variety*  $\mathcal{X}_M$  is shown to detect projectivity when  $\mathfrak{g}$  is a simple classical Lie superalgebra in [18, Theorem 3.4] in the sense that  $M$  is a projective module in  $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$  if and only if  $\mathcal{X}_M = \{0\}$ . When  $M$  is finite dimensional, the case of interest in this paper,  $\mathcal{X}_M$  is also a Zariski closed  $G_{\overline{0}}$ -invariant subcone of  $\mathcal{X}$ .

Furthermore, the associated varieties satisfy certain desirable properties as proven in [18]. Recall that for a super vector space  $V = V_{\overline{0}} \oplus V_{\overline{1}}$ , we define the superdimension of  $V$  to be  $\text{sdim}(V) := \dim V_{\overline{0}} - \dim V_{\overline{1}}$ .

**Proposition 2.3** (Dufflo-Serganova). *Let  $M, N \in \mathcal{F}$  and  $x \in \mathcal{X}$ . Then*

- (1) *if  $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\overline{0}})} M_0$  for some  $\mathfrak{g}_{\overline{0}}$ -module  $M_0$ , then  $\mathcal{X}_M = \{0\}$ ;*
- (2) *for the trivial module  $k$  concentrated in even or odd degree,  $\mathcal{X}_k = \{0\}$ ;*
- (3)  $\mathcal{X}_{M \oplus N} = \mathcal{X}_M \cup \mathcal{X}_N$ ;
- (4)  $\mathcal{X}_{M \otimes N} = \mathcal{X}_M \cap \mathcal{X}_N$ ;
- (5)  $\mathcal{X}_M = \mathcal{X}_{M^*}$ ;
- (6)  $\text{sdim}(M) = \text{sdim}(M_x)$ .

The variety  $\mathcal{X}$  may also be related to the combinatorial invariants in the previous section by the following. Let  $S$  denote all sets of isotropic, pairwise orthogonal roots  $\alpha$ , and let

$$S_m = \{A \in S \mid |A| = m\}.$$

Then  $S_m$  is nonempty for  $1, 2, \dots, \text{def}(\mathfrak{g})$ . It is shown in [18] that for any  $x \in \mathcal{X}$  the  $G_{\overline{0}}$  orbit of  $x$  contains an element of the form  $x_1 + \dots + x_m$  where  $x_i \in \mathfrak{g}_{\alpha_i}$  where  $\{\alpha_1, \dots, \alpha_m\} \in S$  and that this number  $m$  only depends on the orbit. Then define  $m$  to be the *rank* of  $x$  which is denoted  $\text{rank}(x)$ .

Note that the fiber  $M_x$  has a module structure over a much larger superalgebra defined in [18, Section 6], denoted  $\mathfrak{g}_x$ . However,  $\mathfrak{g}_x$  is too large for consideration here. For example, if  $\mathfrak{g} = \mathfrak{gl}(m|n)$ , and  $x$  has rank  $k$  then  $\mathfrak{g}_x \cong \mathfrak{gl}(m-k|n-k)$ . Thus, we always consider  $M_x$  simply as an  $\langle x \rangle$ -module.

**2.3. Constant Jordan Type in Finite Group Schemes.** In [8], for a field  $k$  of characteristic  $p > 0$ , Carlson, Friedlander, and Pevtsova use the simplicity of the representation theory of  $k[t]/t^p$ -modules and a number of cohomological results define and study what is called the Jordan type of a  $k[t]/t^p$ -module and eventually the Jordan type of a module for a finite group scheme if the module satisfies certain conditions.

If we fix a basis for  $M$  a  $k[t]/t^p$ -module of dimension  $n$ , then the associated representation  $\rho : k[t]/t^p \rightarrow M_n(k)$  is completely determined by  $\rho(t)$  which is some  $p$ -nilpotent  $n \times n$  matrix. Furthermore, by a change of basis we can write  $\rho(t)$  in Jordan canonical form. So up to isomorphism, the structure of  $M$  is completely determined by the sizes of the Jordan blocks of  $\rho(t)$  which are necessarily of size  $\leq p$ , i.e. a  $p$ -restricted partition of  $n$ . Thus we may associate to  $\rho$  a *Jordan type*,  $a_1[1] + a_2[2] + \cdots + a_{p-1}[p-1] + a_p[p]$  where  $a_i$  denotes the number of blocks of size  $i$  in the partition of  $n$  associated to  $M$ .

For a finite group scheme  $G$ , the authors consider maps of  $K$ -algebras  $\alpha : K[t]/t^p \rightarrow KG$  which satisfies certain conditions, where  $K/k$  is a field extension. There is an equivalence condition placed on these maps and each equivalence class is referred to as a  $\pi$ -point. The space of  $\pi$ -points admits a scheme structure which is isomorphic to  $\text{Proj } H^\bullet(G, k)$ . If  $M$  is a finite dimensional  $kG$ -module, then restricting along  $\alpha$  allows us to define the Jordan type of  $\alpha^*(M)$  at a  $\pi$ -point  $\alpha$ , and if the Jordan type is the same over all  $\pi$ -points, then  $M$  is said to be of *constant Jordan type*.

### 3. CONSTANT JORDAN TYPE

Modules of constant Jordan type for finite group schemes are studied extensively in [8] and applied in [1], [3], and [9] as well as in numerous others. With this motivation and the observation that  $U(\langle x \rangle) \cong k[t]/t^2$  for  $x \in \mathcal{X}$ , we begin defining an analogous theory for Lie superalgebras.

**3.1. Definitions and Conventions.** Let  $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$  be a finite dimensional Lie superalgebra such that  $\mathcal{X}$  spans  $\mathfrak{g}_{\bar{1}}$ . While the following definitions make sense for arbitrary Lie superalgebras (when  $\mathcal{X} \neq \{0\}$ ), many of the properties and results rely on using the imposed restrictions on  $\mathfrak{g}$ , particularly the use of [18, Theorem 3.4].

The condition imposed is not very restrictive considering  $\mathcal{X}$  spans  $\mathfrak{g}_{\bar{1}}$  for all simple classical Lie superalgebras except  $\mathfrak{osp}(1|2n)$  in which case  $\mathcal{X} = \{0\}$  and all finite dimensional modules are projective trivializing the theory. It is clear then that the spanning property is preserved under products of Lie superalgebras as well giving further instances of application.

By taking advantage of the explicit description of all indecomposable  $U(\langle x \rangle)$ -modules we make the following definitions.

**Definition 3.1.** Let  $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$  and let  $\mathcal{X}$  be the cone of self commuting elements of  $\mathfrak{g}$ . For  $x \in \mathcal{X} \setminus \{0\}$ ,

$$M|_{\langle x \rangle} \cong k_{ev}^{\oplus a_{ev}} \oplus k_{od}^{\oplus a_{od}} \oplus P^{\oplus a_2},$$

where  $P$  is, up to parity change, the unique projective indecomposable  $U(\langle x \rangle)$ -module, and by definition  $a_{ev} + a_{od} + 2a_2 = \dim(M)$ . Then we then define the *super Jordan type* of  $M$  at  $x$  to be the isomorphism type of  $M|_{\langle x \rangle}$  and denote it by  $(a_{ev}|a_{od})[1] + a_2[2]$ .

When there is no need to distinguish the particular dimensions  $a_{ev}$  and  $a_{od}$ , then we set  $a_1 = a_{ev} + a_{od}$  and define the *Jordan type* of  $M$  at  $x$  to be the isomorphism type of

$$M|_{\langle x \rangle} \cong k^{\oplus a_1} \oplus P^{\oplus a_2}$$

and denote it by  $a_1[1] + a_2[2]$ .

**Definition 3.2.** Let  $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}$  and let  $\mathcal{X}$  be the cone of self commuting elements of  $\mathfrak{g}$ . We say that  $M$  is of *constant (super) Jordan type* if the (super) Jordan type of  $M$  at  $x$  is the same for all  $x \in \mathcal{X} \setminus \{0\}$ .

When  $M$  is of constant (super) Jordan type, we define the *(super) Jordan type of  $M$*  to be the (super) Jordan type of  $M$  at  $x$  for any point  $x \in \mathcal{X} \setminus \{0\}$ .

*Remark.* Because the super Jordan type of a module  $M$ , either at a point or for all of  $M$ , is indexed by the numbers  $a_{ev}$ ,  $a_{od}$ , and  $a_2$  and we can express them as an equation  $a_{ev} + a_{od} + a_2 = \dim(M) = n$ , where  $n$  is given by an intrinsic property of  $M$ , we may refer simply refer to the super Jordan type as  $(a_{ev}|a_{od})[1]$  and the Jordan type as  $a_1[1]$  or just the single non-negative integer  $a_1$ . In the language of [8], if  $M$  and  $N$  have super Jordan types  $(a_{ev}|a_{od})[1] + a_2[2]$  and  $(b_{ev}|b_{od})[1] + b_2[2]$  these are called *stably equivalent* if  $a_{ev} = b_{ev}$  and  $a_{od} = b_{od}$  because in the stable category of  $\langle x \rangle$ -supermodules,  $[M] = [N]$  and  $(a_{ev}|a_{od})$  ( $a_1$  respectively) is called the *stable super Jordan type* (*stable Jordan type* respectively) of  $M$ .

An interesting consequence of the additional information of the super dimension of a module is that there is no distinction between modules of constant super Jordan type and modules of constant Jordan type, as seen in the corollary following the lemma.

**Lemma 3.3.** *Let  $M \in \mathcal{F}$  and  $x \in \mathcal{X} \setminus \{0\}$ . Then  $M_x \cong \Omega^0(M|_{\langle x \rangle})$  as  $\langle x \rangle$ -supermodules.*

*Proof.* Because of the explicit description of  $\langle x \rangle$ -supermodules, we can see that when viewing multiplication by  $x$  as a linear operator on  $M$ ,

$$\begin{aligned} \text{Ker}(x) &= \text{Soc}(M|_{\langle x \rangle}) = k_{ev}^{\oplus a_{ev}} \oplus k_{od}^{\oplus a_{od}} \oplus \text{Soc}(P)^{\oplus a_2} \\ \text{Im}(x) &= \text{Soc}(P)^{\oplus a_2} \end{aligned}$$

where  $M|_{\langle x \rangle} \cong k_{ev}^{\oplus a_{ev}} \oplus k_{od}^{\oplus a_{od}} \oplus P^{\oplus a_2}$  is the Jordan type of  $M$  at  $x$ . Then it is clear that  $M_x \cong k_{ev}^{\oplus a_{ev}} \oplus k_{od}^{\oplus a_{od}}$  as an  $\langle x \rangle$ -supermodule, and the claim is proven.  $\square$

**Corollary 3.4.** *Let  $M \in \mathcal{F}$  and  $x \in \mathcal{X} \setminus \{0\}$ . Then*

- (1) *in the stable module category of  $\langle x \rangle$ -supermodules,  $[M_x] = [M|_{\langle x \rangle}]$ ;*
- (2) *the stable Jordan type of  $M$  at  $x$  is  $a_1 = \dim(M_x)$  and  $M$  is of constant Jordan type if and only if this equation holds for any  $x \in \mathcal{X} \setminus \{0\}$ ;*
- (3) *if  $M$  has stable Jordan type  $a_1$  at a point  $x$ , then the stable super Jordan type is uniquely determined by  $\text{sdim}(M)$ ;*
- (4)  *$M$  is of constant Jordan type if and only if  $M$  is of constant super Jordan type.*

*Proof.* Parts (1) and (2) follow immediately from Lemma 3.3. For (3), recall Proposition 2.3 (6). Since  $\text{sdim}(M) = \text{sdim}(M_x)$  is independent of the choice of  $x$  and stable the Jordan type of  $M$  is  $a_1 = \dim(M_x)$ , then we have  $a_{ev} + a_{od} = \dim(M_x)$  and  $a_{ev} - a_{od} = \text{sdim}(M_x)$ , and thus

$$a_{ev} = \frac{\dim(M_x) + \text{sdim}(M_x)}{2} \quad \text{and} \quad a_{od} = \frac{\dim(M_x) - \text{sdim}(M_x)}{2}$$

for any  $x \in \mathcal{X} \setminus \{0\}$ .

Then part (4) follows directly from part (3).  $\square$

**Convention.** Given Corollary 3.4 (4), we will only use the term modules of constant Jordan type since the “super” condition follows automatically.

**3.2. Properties of Modules of Constant Jordan Type.** We now seek to identify and understand how these modules behave under standard categorical operations. With the goal of identification, we observe the following properties of these modules.

**Proposition 3.5.** *Let  $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}$  be a module of constant Jordan type. Then either  $\mathcal{X}_M = \{0\}$  or  $\mathcal{X}_M = \mathcal{X}$ . Additionally,  $\mathcal{X}_M = \{0\}$  is equivalent to the statement that  $M$  is projective or that  $M$  has stable Jordan type  $a_1 = 0$ .*

*Furthermore, if  $\mathfrak{g}$  is simple basic classical and  $M$  is indecomposable and is not projective, then  $M$  lies in a block of maximal atypicality, i.e.,  $M$  has atypicality  $d = \text{def}(\mathfrak{g})$ .*

*Proof.* If  $M$  has stable Jordan type  $a_1$  then by Corollary 3.4 (2),  $a_1 = \dim(M_x)$  for all  $x \in \mathcal{X}$ . Then by the definition of  $\mathcal{X}_M$ , if  $a_1 = 0$  then  $\mathcal{X}_M = \{0\}$ , and  $\mathcal{X}_M = \mathcal{X}$  otherwise.

By [18, Theorem 3.4],  $M$  is projective in  $\mathcal{F}$  if and only if  $\mathcal{X}_M = \{0\}$  and this is equivalent to  $M$  having stable Jordan type  $a_1 = 0$  by definition.

Let  $\mathcal{X}_k = \{x \in \mathcal{X} \mid \text{rank}(x) = k\}$ . Then  $\bar{\mathcal{X}}_k = \bigcup_{i \leq k} \mathcal{X}_i$  and  $\bar{\mathcal{X}}_d = \mathcal{X}$  where  $d$  is the defect of  $\mathfrak{g}$ . Then by [18, Theorem 5.3],  $\mathcal{X}_M \subseteq \bar{\mathcal{X}}_\ell$  where  $M$  has atypicality  $\ell$ . Since  $M$  is not projective by assumption,  $\mathcal{X}_M \neq \{0\}$ , and so it must be that  $\mathcal{X}_M = \mathcal{X}$  which implies that  $\ell = d$ .  $\square$

**Lemma 3.6.** *Let  $Q \in \mathcal{F}$ . Then  $Q$  is projective in  $\mathcal{F}$  if and only if  $Q|_{\langle x \rangle}$  is projective for all  $x \in \mathcal{X}$ .*

*Proof.* Since  $Q$  is projective if and only if  $\mathcal{X}_Q = \{0\}$  and the alternative definition of  $\mathcal{X}_Q$  in Equation 2.2 implies that this is equivalent to the condition that  $Q|_{\langle x \rangle}$  is projective for all  $x \in \mathcal{X}$ .  $\square$

In the following proposition, in parts (1)–(4), there are straightforward dimension formulas (in addition to the ones for super dimension given) and so the super Jordan type of the modules is completely determined. However, in part (5) there is no general formula for determining  $\dim(\Omega^n(M))$  and so we may only determine the super Jordan type of  $\Omega^n(M)$  up to stable equivalence.

**Proposition 3.7.** *Let  $M, N \in \mathcal{F}$  be modules of constant Jordan type  $a_1[1] + a_2[2]$  and  $b_1[1] + b_2[2]$  respectively. Then*

- (1)  $M \oplus N$  is of constant Jordan type  $(a_1 + b_1)[1] + (a_2 + b_2)[2]$  and  $\text{sdim}(M \oplus N) = \text{sdim}(M) + \text{sdim}(N)$ ;
- (2)  $M^*$  is of constant Jordan type  $a_1[1] + a_2[2]$  and  $\text{sdim}(M^*) = \text{sdim}(M)$ ;
- (3)  $M \otimes N$  is of constant Jordan type  $(a_1 \cdot b_1)[1] + (a_1 \cdot b_2 + a_2 \cdot b_1 + a_2 \cdot b_2)[2]$  and  $\text{sdim}(M \otimes N) = \text{sdim}(M) \cdot \text{sdim}(N)$ ;
- (4)  $\text{Hom}_k(M, N)$  is of constant Jordan type  $(a_1 \cdot b_1)[1] + (a_1 \cdot b_2 + a_2 \cdot b_1 + a_2 \cdot b_2)[2]$  and  $\text{sdim}(\text{Hom}_k(M, N)) = \text{sdim}(M) \cdot \text{sdim}(N)$ ;
- (5)  $\Omega^n(M)$  is of stable Jordan type  $a_1$  for all  $n \in \mathbb{Z}$  and  $\text{sdim}(\Omega^n(M)) = (-1)^n \text{sdim}(M)$ .

*Proof.* By [31, Proposition 3.5 (g)], and Lemma 3.3, since  $\dim(M_x) = a_1$  and  $\dim(N_x) = b_1$  for any  $x \in \mathcal{X} \setminus \{0\}$ , then  $\dim((M \oplus N)_x) = \dim(\Omega^0((M \oplus N)|_{\langle x \rangle})) = a_1 + b_1$  which shows (1).

For (2),  $M^* = \text{Hom}_k(M, k_{ev})$ , so then

$$\begin{aligned} M^*|_{\langle x \rangle} &= \text{Hom}_k(M, k_{ev})|_{\langle x \rangle} \cong \text{Hom}_k(M|_{\langle x \rangle}, k_{ev}) \cong \text{Hom}_k(k^{\oplus a_1} \oplus P^{\oplus a_2}, k_{ev}) \\ &\cong \text{Hom}_k(k^{\oplus a_1}, k_{ev}) \oplus \text{Hom}_k(P^{\oplus a_2}, k_{ev}) \cong k^{\oplus a_1} \oplus P^{\oplus a_2} \end{aligned}$$



and so if  $M$  has super Jordan type  $(a_{ev}|a_{od})[1]$  at  $x$  then  $M^*$  also has super Jordan type  $(a_{ev}|a_{od})[1]$  and  $\text{sdim}(M_x) = \text{sdim}(M_x^*)$ . This is again constant over all  $x$  and so  $M^*$  has constant Jordan type  $a_{ev} + a_{od} = a_1$ .

By [31, Proposition 3.5 (f)] and again using Lemma 3.3,  $(M \otimes N)_x = \Omega^0((M \otimes N)|_{\langle x \rangle})$  as  $\langle x \rangle$ -modules. Then  $[(M \otimes N)|_{\langle x \rangle}] = [k^{\oplus a_1} \otimes k^{\oplus b_1}]$  and so  $\dim((M \otimes N)_x) = a_1 \cdot b_1$  for any  $x \in \mathcal{X} \setminus \{0\}$  and  $M \otimes N$  is of constant Jordan type  $a_1 \cdot b_1$  and (3) is established.

Part (4) follows directly from (2) and (3) by the canonical isomorphism  $\text{Hom}_k(M, N) \cong N \otimes M^*$ .

Finally, for (5) let

$$0 \longrightarrow \Omega^1(M) \longrightarrow Q \longrightarrow M \longrightarrow 0$$

be the exact sequence which defines  $\Omega^1(M)$ . Then  $Q|_{\langle x \rangle}$  is the projective by Lemma 3.6 and restriction is an exact functor, so

$$0 \longrightarrow \Omega^1(M)|_{\langle x \rangle} \longrightarrow Q|_{\langle x \rangle} \longrightarrow M|_{\langle x \rangle} \longrightarrow 0$$

is exact as well. Thus,  $\Omega^1(M)|_{\langle x \rangle} \cong \Omega^1(M|_{\langle x \rangle}) \oplus R$  where  $R$  is a projective  $\langle x \rangle$ -module. Since  $M|_{\langle x \rangle} \cong k^{\oplus a_1} \oplus P^{\oplus a_2}$  and syzygies commute with direct sums and are trivial on projective summands, we have that

$$\Omega^1(M)|_{\langle x \rangle} \cong \Omega^1(k^{\oplus a_1}) \oplus R \cong \Pi(k^{\oplus a_1}) \oplus R.$$

Thus  $\Omega^1(M)$  has constant super Jordan type  $(a_{od}|a_{ev})[1]$  and Jordan type  $a_1$ . A dual argument shows the same for  $\Omega^{-1}(M)$  and induction shows that  $\Omega^n(M)$  is of constant Jordan type  $a_1$  for any  $n \in \mathbb{Z}$  and  $\text{sdim}(\Omega^n(M)) = (-1)^n \text{sdim}(M)$ .  $\square$

The previous proposition gives many ways of constructing new modules of constant Jordan type from already known modules of constant Jordan type. In fact for any choice of  $\mathfrak{g}$ , because the Jordan types in this setting are so simple, it trivializes the question of which Jordan types are realized up stable equivalence because we may take  $M = k^n$  for  $n \in \mathbb{N}$  and chose the concentration of the trivial summands to realize any stable Jordan type. Thus the question of realization must be modified to include indecomposability or to consider the full Jordan type.

The definition of Jordan type depends *a priori* on checking the Jordan type of  $M$  at all points  $x \in \mathcal{X}$  of which there are infinitely many. However, when  $\mathfrak{g}$  is a basic classical Lie superalgebra with indecomposable Cartan matrix, [18, Theorem 4.2] shows that there are finitely many  $G_{\overline{0}}$ -orbits in  $\mathcal{X}$  and the Jordan type is invariant along the orbit since the modules in the same orbit are isomorphic via twisting by the adjoint action. This immediately proves the following theorem. See [18, Remark 4.1] for more on which Lie superalgebras satisfy the condition (which includes the simple basic classical types).

**Theorem 3.8.** *Let  $\mathfrak{g}$  be a basic classical Lie superalgebra with indecomposable Cartan matrix. Then  $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$  is of constant Jordan type if and only if the Jordan type of  $M$  is constant over a finite number of a finite set of points which are orbit representatives of the  $G_{\overline{0}}$  action on  $\mathcal{X}$ .*

**3.3. Examples.** We consider some examples and non-examples here which further illustrate the structure and properties of modules of constant Jordan type.

**Example 3.9.** Since  $k_{ev}$  and  $k_{od}$  are of constant Jordan type  $1[1]$ , in cases when these modules have nonzero complexity over  $\mathfrak{g}$ , by Proposition 3.7 (5),  $\Omega^n(k)$  give infinitely many nonisomorphic modules of stable Jordan type  $1[1]$  as well. The reader who is familiar with endotrivial modules will note that each of these syzygies is endotrivial as well. This topic is discussed further in Section 4, and Section 3.4 gives a construction of an example of modules of constant Jordan type which are not endotrivial.

The next examples show that, unfortunately, modules of constant Jordan type are not closed under taking direct summands.

**Example 3.10.** Let  $\mathfrak{g} = \mathfrak{sl}(1|1)$  where  $\mathfrak{sl}(1|1)_{\bar{0}} = \{t\}$  and  $\mathfrak{sl}(1|1)_{\bar{1}} = \{x, y\}$  are bases for their respective components and the only nontrivial bracket is  $[x, y] = xy + yx = t$ . Then one can verify directly that  $\mathcal{X} = k \cdot x \cup k \cdot y$  in this case. Then under the action of  $G_{\bar{0}} \cong k$  (see the remark at the end of the previous section), there are two orbits whose representatives are chosen to be  $x$  and  $y$  for simplicity.

Let  $K(0)$  and  $K^-(0)$  be the Kac and dual Kac module of the trivial module  $k_{ev}$ , as defined in [6, Section 3.1]. Then  $K(0)$  and  $K^-(0)$  are both two dimensional modules with a one dimensional submodule as the socle and a one dimensional head. The structure of the modules is given by

$$K(0) : \begin{array}{c} k_{ev} \\ | \\ y \\ | \\ k_{od} \end{array} \qquad K^-(0) : \begin{array}{c} k_{od} \\ | \\ x \\ | \\ k_{ev} \end{array}$$

where  $t$  acts trivially on all of  $K(0)$  and  $K^-(0)$ . Then  $K(0)$  is projective as a  $\langle y \rangle$ -module and  $K^-(0)$  is projective as a  $\langle x \rangle$ -module and  $\mathcal{X}_{K(0)} = k \cdot x$  and  $\mathcal{X}_{K^-(0)} = k \cdot y$ , and so neither module is of constant Jordan type by Proposition 3.5. However, we see that  $\mathcal{X}_{K(0) \oplus K^-(0)} = k \cdot x \cup k \cdot y$  and in fact  $K(0) \oplus K^-(0)$  is a module of constant Jordan type  $2[1] + 1[2]$ .

**Example 3.11.** Furthermore, considering the setting of the last example, define

$$M = k_{ev} \oplus K(0) \quad \text{and} \quad N = k_{ev} \oplus K^-(0).$$

Then we have that both  $\mathcal{X}_M = \mathcal{X}$  and  $\mathcal{X}_N = \mathcal{X}$  and  $M \oplus N$  is of constant Jordan type  $4[1] + 1[2]$ , but neither  $M$  nor  $N$  is of constant Jordan type. This shows that even in a situation with the possibly stronger assumption that the module summands have maximal associated varieties, modules of constant Jordan type are not closed under direct summands.

**Example 3.12.** Again, let  $\mathfrak{g} = \mathfrak{sl}(1|1)$  and define  $M$  and  $N$  to be four dimensional modules with structures given by

$$M : \begin{array}{ccccc} & & k_{ev} & & k_{ev} \\ & x & / & y & / \\ k_{od} & & & & k_{od} \end{array} \qquad N : \begin{array}{ccccc} k_{ev} & & k_{ev} & & \\ & y & & x & \\ & / & & / & \\ & k_{od} & & k_{od} & \end{array}$$

and all other actions are zero. Then  $M \oplus N$  is of constant Jordan type  $2[1] + 3[2]$ , and  $M$  and  $N$  are indecomposable, but neither  $M$  nor  $N$  is of constant Jordan type, although we do note that  $\mathcal{X}_M = k \cdot y$  and  $\mathcal{X}_N = k \cdot x$  in this case.

These examples show that these modules are not closed under taking direct summands because modules may decompose into summands which are not of constant Jordan type themselves, but are symmetric in such a way that they become constant Jordan type when summed together. Note that this situation is contrasted with that of [8] in this respect.

We conclude this section with an example that shows that there are indecomposable modules of stable Jordan type other than 1, which even in simple cases is non-trivial.

**3.4. A Nontrivial Example.** We begin by considering a proposition which will be useful in constructing the example of interest in this section.

**Proposition 3.13.** *Let  $\mathfrak{g}$  be a Lie superalgebra and let  $\mathfrak{h}$  be a Lie subalgebra such that for each projective module  $Q$  in  $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}^{\overline{0}})}$ ,  $Q|_{\mathfrak{h}}$  is projective in  $\mathcal{F}_{(\mathfrak{h}, \mathfrak{h}^{\overline{0}})}$ . Let  $k \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}^{\overline{0}})}$  be the trivial module concentrated in either degree and denote the decomposition by restriction as  $\Omega_{\mathfrak{g}}^n(k)|_{\mathfrak{h}} \cong \Omega_{\mathfrak{h}}^n(k) \oplus P_n$  for all  $n \in \mathbb{Z}$  where  $P_n$  is projective in  $\mathcal{F}_{(\mathfrak{h}, \mathfrak{h}^{\overline{0}})}$ . Then there are extensions*

$$\begin{aligned} 0 \rightarrow P_n \rightarrow \Omega_{\mathfrak{g}}^n(k) \rightarrow \Omega_{\mathfrak{h}}^n(k) \rightarrow 0 & \quad \text{for } n \geq 0 \\ 0 \rightarrow \Omega_{\mathfrak{h}}^n(k) \rightarrow \Omega_{\mathfrak{g}}^n(k) \rightarrow P_n \rightarrow 0 & \quad \text{for } n \leq 0 \end{aligned}$$

as modules in  $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}^{\overline{0}})}$ .

*Proof.* We use an inductive argument as follows, noting that when  $n = 0$ , we have  $k \cong \Omega_{\mathfrak{g}}^0(k) \cong \Omega_{\mathfrak{h}}^0(k)$  and so  $P = 0$  and the claim is trivial. We proceed by induction and include the proof for the  $n \geq 0$  case and the  $n \leq 0$  proof is dual to the following.

Recall that  $\Omega^{n+1}(k)$  is by definition  $\Omega^1(\Omega^n(k))$  and we have the following two exact sequences which define  $\Omega_{\mathfrak{g}}^{n+1}(k)$  and  $\Omega_{\mathfrak{h}}^{n+1}(k)$

$$\begin{array}{ccccccc} & & \text{Ker}(\pi_{n+1}) & & P_n & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_{\mathfrak{g}}^{n+1}(k) & \longrightarrow & P_{\mathfrak{g}} & \xrightarrow{\psi_{\mathfrak{g}}} & \Omega_{\mathfrak{g}}^n(k) \longrightarrow 0 \\ & & \downarrow \pi_{n+1} & & \downarrow \pi_n & & \\ 0 & \longrightarrow & \Omega_{\mathfrak{h}}^{n+1}(k) & \longrightarrow & P_{\mathfrak{h}} & \xrightarrow{\psi_{\mathfrak{h}}} & \Omega_{\mathfrak{h}}^n(k) \longrightarrow 0 \end{array}$$

where, both the surjection given by  $\pi_n$  and its kernel, exist by the inductive hypothesis. Note that this is in fact an isomorphism when  $n = 0$  and so  $P_0 = 0$ . Additionally, the map  $\pi_n \circ \psi_{\mathfrak{g}}$  can be lifted to a map  $\pi_{n+1}$  since  $\psi_{\mathfrak{h}}$  is surjective. Furthermore,  $\pi_{n+1}$  can be seen to be surjective by considering the commutative square restricted to  $\mathfrak{h}$

$$\begin{array}{ccc} P_{\mathfrak{h}} \oplus P^{\perp} & \xrightarrow{\psi_{\mathfrak{h}} \oplus \psi^{\perp}} & \Omega_{\mathfrak{h}}^n(k) \oplus P_n \\ \downarrow \pi_{n+1} & & \downarrow \pi_n \\ P_{\mathfrak{h}} & \xrightarrow{\psi_{\mathfrak{h}}} & \Omega_{\mathfrak{h}}^n(k) \end{array}$$

and so  $\pi_n$  is just projection onto  $\Omega_{\mathfrak{h}}^n(k)$ ,  $\pi_{n+1}$  is projection onto  $P_{\mathfrak{h}}$ , and  $\psi^{\perp}$  gives a surjection of  $P^{\perp} = \text{Ker}(\pi_{n+1})$  onto  $P_n$  as  $\mathfrak{g}$ -modules.

Since  $\psi_{\mathfrak{g}}(\Omega_{\mathfrak{g}}^{n+1}(k)) = 0$ , then  $\psi_{\mathfrak{h}}(\pi_{n+1}(\Omega_{\mathfrak{g}}^{n+1}(k))) = 0$  and so  $\pi_{n+1}(\Omega_{\mathfrak{g}}^{n+1}(k)) \subseteq \Omega_{\mathfrak{h}}^{n+1}(k)$ . However, by again recalling that  $\pi_{n+1}$  is surjective and considering the restriction to  $\mathfrak{h}$  above,

we see that in fact  $\pi_{n+1}(\Omega_{\mathfrak{g}}^{n+1}(k)) = \Omega_{\mathfrak{h}}^{n+1}(k)$  and that the kernel of this map is given by the pullback of  $\Omega_{\mathfrak{g}}^{n+1}(k)$  and  $\text{Ker}(\pi_{n+1})$  mapping into  $P_{\mathfrak{g}}$  and which we denote  $P_{n+1}$ . Thus the diagram can be completed to

$$\begin{array}{ccccccc}
0 & \longrightarrow & P_{n+1} & \longrightarrow & \text{Ker}(\pi_{n+1}) & \xrightarrow{\psi^\perp} & P_n \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Omega_{\mathfrak{g}}^{n+1}(k) & \longrightarrow & P_{\mathfrak{g}} & \xrightarrow{\psi_{\mathfrak{g}}} & \Omega_{\mathfrak{g}}^n(k) \longrightarrow 0 \\
& & \downarrow \pi_{n+1} & & \downarrow \pi_{n+1} & & \downarrow \pi_n \\
0 & \longrightarrow & \Omega_{\mathfrak{h}}^{n+1}(k) & \longrightarrow & P_{\mathfrak{h}} & \xrightarrow{\psi_{\mathfrak{h}}} & \Omega_{\mathfrak{h}}^n(k) \longrightarrow 0
\end{array}$$

which yields the exact sequence  $0 \rightarrow P_{n+1} \rightarrow \Omega_{\mathfrak{g}}^{n+1}(k) \rightarrow \Omega_{\mathfrak{h}}^{n+1}(k) \rightarrow 0$  inductively from the previous one. Note that  $P_{n+1}$  is projective when restricted to  $\mathcal{F}_{(\mathfrak{h}, \mathfrak{h}\overline{\mathfrak{g}})}$  since the other two modules on the top row are as well and that by construction,  $\Omega_{\mathfrak{g}}^{n+1}(k)|_{\mathfrak{h}} \cong \Omega_{\mathfrak{h}}^{n+1}(k) \oplus P_{n+1}$  as desired.  $\square$

**Corollary 3.14.** *Given the above assumptions with the decomposition by restriction denoted as  $\Omega_{\mathfrak{g}}^n(k)|_{\mathfrak{h}} \cong \Omega_{\mathfrak{h}}^n(k) \oplus P_n$ , then*

$$\begin{aligned}
\text{Hd}(\Omega_{\mathfrak{h}}^n(k)) &\subseteq \text{Hd}(\Omega_{\mathfrak{g}}^n(k)) && \text{for } n \geq 0 \\
\text{Soc}(\Omega_{\mathfrak{h}}^n(k)) &\subseteq \text{Soc}(\Omega_{\mathfrak{g}}^n(k)) && \text{for } n \leq 0.
\end{aligned}$$

Now we consider the case of  $\Omega^n(k) \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}\overline{\mathfrak{g}})}$  where  $\mathfrak{g}$  is the detecting subalgebra  $\mathfrak{f}_r = \mathfrak{sl}(1|1) \times \cdots \times \mathfrak{sl}(1|1)$  and there are  $r$  copies of  $\mathfrak{sl}(1|1)$ . Then  $\Omega^n(k)/\text{Soc}(\Omega^n(k))$  for  $n > 0$  and  $\text{Rad}(\Omega^n(k))$  for  $n < 0$  are of constant Jordan type but are decomposable when  $r = 1$ . When  $r > 1$  we show that for  $n = 1$  the module is indecomposable and conjecture that this holds for all  $n \neq 0$ . Furthermore,  $\Omega^n(k)/\text{Soc}(\Omega^n(k))$  and  $\text{Rad}(\Omega^{-n}(k))$  have the same Jordan type for  $n \geq 0$  and are in fact isomorphic since the projective indecomposables in this block are self dual (up to parity change).

This can be seen by considering the projective covers of the simple modules in the principal block, where indecomposable modules of constant Jordan type necessarily exist, shown later in Lemma 5.1. Since the only simple modules are  $k_{ev}$  and  $k_{od}$ , and the induced modules  $P(0) = U(\mathfrak{f}) \otimes_{U(\mathfrak{f}\overline{\mathfrak{g}})} k$  have one dimensional simple heads and simple socles, these induced modules are the projective indecomposable modules in the block.

As in [31, Section 5.2], we observe that in the principal block, we may simplify the situation from considering  $U(\mathfrak{f}_r)$ -modules to modules over a super symmetric algebra generated by odd elements. So let  $V(\mathfrak{a}) := S^{\text{sup}}(\mathfrak{a}) \cong \Lambda(\mathfrak{a})$  where  $\mathfrak{a} = (\mathfrak{f}_r)_{\overline{1}}$  and has a fixed basis of  $\{x_1, \dots, x_r, y_1, \dots, y_r\}$ . This is because all even elements commute and in the principal block, they act by zero, allowing this reduction. Let  $V(\mathfrak{a}_s)$  denote an exterior algebra with  $s$  elements of degree 1. Also note that under this equivalence we have

$$U(\mathfrak{f}) \otimes_{U(\mathfrak{f}\overline{\mathfrak{g}})} k \cong V(\mathfrak{a}_{2r}) \quad \text{and} \quad \mathcal{X} = \sum_{i=1}^r a_i x_i + b_i y_i$$

where at most one of the  $a_i$  and  $b_i$  are nonzero for each  $1 \leq i \leq r$ . However, an arbitrary element of  $V(\mathfrak{a}_{2r})$  squares to zero, and given the results of Section 5, it will be meaningful

to consider this example where  $\mathcal{X}$  is given by

$$\mathcal{X} = \sum_{i=1}^r a_i x_i + b_i y_i$$

where arbitrary coefficients are allowed.

Now we examine  $\Omega^n(k)/\text{Soc}(\Omega^n(k))$  when  $r > 1$  and  $n > 0$ . Recall that  $\Omega^n(k)$  is of stable Jordan type 1[1] and since we only have information up to stable equivalence, the projective summands in the Jordan decompositions will be generically denoted as  $P_M$  for a module  $M$ . In the decomposition

$$\Omega^n(k)|_{\langle x \rangle} \cong k \oplus P_{\Omega^n(k)}$$

for  $x \in \mathcal{X} \setminus \{0\}$ , the trivial summand must appear in the head of  $\Omega^n(k)$  by Corollary 3.14.

Since the socle of  $\Omega^n(k)$  is contained in the socle of the decomposition  $M|_{\langle x \rangle} \cong k \oplus P_M$  and  $k \cap \text{Soc}(\Omega^n(k)) = \emptyset$ , then  $\text{Soc}(\Omega^n(k)) \subseteq \text{Soc}(P_M)$ . Then taking  $\Omega^n(k)/\text{Soc}(\Omega^n(k))$  yields  $\dim(\text{Soc}(\Omega^n(k)))$  new  $k$  summands in each decomposition of  $M|_{\langle x \rangle}$  and so  $\Omega^n(k)/\text{Soc}(\Omega^n(k))$  is of stable Jordan type  $1 + \dim(\text{Soc}(\Omega^n(k)))$ .

In particular, for any  $\mathfrak{g}$  the projective cover of the trivial module  $k$  has a simple head and a simple socle so  $\Omega^1(k)$  has a simple socle and thus  $\Omega^1(k)/\text{Soc}(\Omega^1(k))$  is a module of stable Jordan type 2[1]. Similarly,  $\text{Rad}(\Omega^{-1}(k))$  has stable Jordan type 2[1].

Additionally, both these modules are indecomposable. The argument for  $\Omega^1(k)/\text{Soc}(\Omega^1(k))$  is given here and the one for  $\text{Rad}(\Omega^{-1}(k))$  is dual. Assume that  $\Omega^1(k)/\text{Soc}(\Omega^1(k))$  decomposes into  $M \oplus N$ . The head of  $\Omega^1(k)/\text{Soc}(\Omega^1(k))$  is isomorphic to  $\Lambda^1(\mathfrak{a})$  and the socle is isomorphic to  $\Lambda^{\dim \mathfrak{a}-1}(\mathfrak{a})$  and these two are distinct since  $r > 1$  by assumption. Then we can write  $\text{Hd}(\Omega^1(k)/\text{Soc}(\Omega^1(k))) \cong \text{Hd}(M) \oplus \text{Hd}(N)$  with bases  $\{m_1, \dots, m_s\}$  and  $\{n_1, \dots, n_t\}$  such that their union is a basis for  $\text{Hd}(\Omega^1(k)/\text{Soc}(\Omega^1(k)))$  (and so  $s + t = 2r$ ). Then since  $m_1 \otimes n_1 = -n_1 \otimes m_1 \neq 0$  and is in the image of both  $\text{Hd}(M)$  and  $\text{Hd}(N)$ , then they have nontrivial intersection unless either  $s$  or  $t$  is 0. Thus,  $\text{Hd}(\Omega^1(k)/\text{Soc}(\Omega^1(k)))$  is indecomposable.

It is possible that  $\Omega^n(k)/\text{Soc}(\Omega^n(k))$  will be indecomposable for all  $n > 0$  but the techniques used here are not sufficient to provide a general proof.

#### 4. ENDOTRIVIAL MODULES

Endotrivial modules are an important and interesting class of module which has been studied in various contexts since Dade introduced the notion in 1978 in modular representation theory ([16], [17]). Other significant progress includes [14], [15], and [27] for representations of  $p$ -groups, [12], [13] for finite group schemes, and [11], [10] for finite groups of Lie type. In general, if  $T(G)$  denotes the set of endotrivial  $G$ -modules in the stable module category (in one of the above contexts), then such modules form a group where the operation is the tensor product.

The author began the study of endotrivial modules for Lie superalgebras in [31] and gave further study in [32]. These papers present classifications of the group of endotrivial modules for detecting subalgebras (as defined in [5]) and for  $\mathfrak{gl}(m|n)$ , respectively.

The following theorem is powerful in that it allows us to determine when a module is endotrivial by checking this property locally, and recalling the remark at the end of Section 3.2, this must be done only at a finite number of points in many cases. The theorem is analogous to [8, Theorem 5.6] and is proven in a similar way.

**Theorem 4.1.** *Let  $\mathfrak{g}$  be a finite dimensional Lie superalgebra such that  $\mathcal{X}$  spans  $\mathfrak{g}_{\overline{1}}$  and  $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$  be a module. Then  $M$  is an endotrivial module if and only if  $M$  is a module of constant Jordan type  $1[1] + m[2]$  for some  $m \geq 0$ .*

*Proof.* First assume that  $M$  is an endotrivial module. Then by definition,

$$M \otimes M^* \cong \text{End}_k(M) \cong k_{ev} \oplus P$$

and since restriction commutes with the above,

$$M|_{\langle x \rangle} \otimes M^*|_{\langle x \rangle} \cong \text{End}_k(M|_{\langle x \rangle}) \cong \text{End}_k(M)|_{\langle x \rangle} \cong k_{ev}|_{\langle x \rangle} \oplus P|_{\langle x \rangle}$$

for any  $x \in \mathcal{X} \setminus \{0\}$ .

By considering the Jordan type of  $M$  at  $x$ ,  $M|_{\langle x \rangle} \cong k^{\oplus a_1} \oplus Q^{\oplus a_2}$ , and recalling Proposition 3.7 (applied at the single point  $x$ ),

$$(4.2) \quad M|_{\langle x \rangle} \otimes M^*|_{\langle x \rangle} \cong k^{\oplus a_1 \cdot a_1} \oplus Q^{\oplus 2(a_1 \cdot a_2 + a_2^2)}$$

it is clear that if  $M$  has stable Jordan type  $a_1[1]$  at  $x$  then  $\text{End}_k(M)$  has stable Jordan type  $a_1^2[1]$  at  $x$ . Since  $\text{End}_k(M)$  has stable Jordan type  $1[1]$  at all points  $x \in \mathcal{X} \setminus \{0\}$ , then  $M$  is of stable Jordan type  $1[1]$ .

Now assume that  $M$  is of stable Jordan type  $1[1]$  and consider the endomorphism algebra of  $M$ . If we fix a basis for  $M$ , we can think of any endomorphism as a  $d \times d$  matrix where  $d = \dim(M)$  and the trace of the endomorphism is independent of the choice of basis. Since the field  $k$  has characteristic 0, there is a homogeneous degree  $\overline{0}$  map

$$\begin{aligned} \alpha : k_{ev} &\rightarrow \text{End}_k(M) \\ c &\mapsto \frac{c}{d} \cdot \text{Id}_M \end{aligned}$$

and the composition  $\text{Tr} \circ \alpha = \text{Id}_k$  and so there is a splitting  $\text{End}_k(M) \cong k_{ev} \oplus \text{Ker}(\text{Tr})$  as modules in  $\mathcal{F}$ .

Since  $M$  has stable Jordan type  $1[1]$ , considering Equation 4.2 it is clear that  $\text{End}_k(M)$  also has stable Jordan type  $1[1]$  as well and in particular,  $\text{Ker}(\text{Tr})|_{\langle x \rangle}$  is projective for all  $x \in \mathcal{X} \setminus \{0\}$  by comparing the projective summands in the two decompositions. Finally, by recalling that  $\mathcal{X}$  detects projectivity ([18, Theorem 3.4]), since  $\text{Ker}(\text{Tr})|_{\langle x \rangle}$  is projective for all  $x$ ,  $\text{Ker}(\text{Tr})$  is projective as a module in  $\mathcal{F}$ . Thus,  $\text{End}_k(M)$  is endotrivial as claimed.  $\square$

The following corollary follows immediately from Proposition 3.5.

**Corollary 4.3.** *Let  $\mathfrak{g}$  be simple basic classical and  $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$  be an endotrivial module. Then  $\mathcal{X}_M = \mathcal{X}$  and  $M$  lies in a block of maximal atypicality.*

This partially recovers an important reduction used in the classification of endotrivial modules for detecting subalgebras in [31, Lemma 5.2] since  $\mathfrak{sl}(1|1)$  is a simple basic classical Lie superalgebra. Additionally, in the proof of that classification, certain conditions are used to describe endotrivial modules and we note here that [31, Section 5.3, condition (2)] identifies a module as stable Jordan type  $1[1]$ .

This corollary can be recovered by the Kac-Wakimoto conjecture in the cases where it is known to hold, i.e.  $\mathfrak{gl}(m|n)$  and  $\mathfrak{osp}(m|2n)$  by [4], [30], and [25]. The conjecture states that for a simple module  $L$ ,

$$\text{sdim}(L) \neq 0 \quad \text{if and only if} \quad \text{atyp}(L) = \text{def}(\mathfrak{g}).$$

**Proposition 4.4.** *Let  $\mathfrak{g}$  be a Lie superalgebra where the Kac-Wakimoto conjecture holds (either  $\mathfrak{gl}(m|n)$  or  $\mathfrak{osp}(m|2n)$ ) and let  $\mathcal{E}_{\mathfrak{g}}$  be the category of finite dimensional integrable  $\mathfrak{g}$ -supermodules, a full subcategory of  $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$ . Let  $M \in \mathcal{E}_{\mathfrak{g}}$  be an endotrivial module. Then  $\text{atyp}(M) = \text{def}(\mathfrak{g})$ , i.e.  $M$  lies in a block of maximal atypicality.*

*Proof.* A corollary of Proposition 3.7 is that  $\text{sdim}(M) \neq 0$  if and only if  $\text{sdim}(M \otimes M^*) \neq 0$  and so if  $M$  is an indecomposable endotrivial module, then  $M \otimes M^* \cong k \oplus P$  and since  $\text{sdim}(P) = 0$ , then we conclude that  $\text{sdim}(M) \neq 0$  and thus  $\text{atyp}(M) = \text{def}(\mathfrak{g})$  by the Kac-Wakimoto conjecture.  $\square$

## 5. CONSTANT JORDAN TYPE FOR TYPE $\mathfrak{f}$ DETECTING SUBALGEBRAS

Recall the detecting subalgebra  $\mathfrak{f}_r = \mathfrak{sl}(1|1) \times \cdots \times \mathfrak{sl}(1|1)$  first mentioned in Section 3.4. In this setting, a natural enlargement of  $\mathcal{X}$  presents itself and allows for more progress to be made in understanding modules of constant Jordan type. In particular, we are able to achieve closure of modules of constant Jordan type under taking direct summands (Proposition 5.2), construct (super) vector bundles on  $\mathbb{P}^{2r-1}$  (Section 5.2.2) and completely classify modules of constant Jordan type over  $\mathfrak{f}_1$  (Theorem 5.9).

**5.1. Enlarging the Self Commuting Cone.** Since this Lie superalgebra is a product (when  $r > 1$ ) it is not simple basic classical and so Proposition 3.5 does not apply. Thus, we begin by showing that for this particular case of interest, non-projective indecomposable modules of constant Jordan type must necessarily exist in the principal block, i.e., the weights of  $M$  are all zero.

**Lemma 5.1.** *Let  $M$  be a non-projective indecomposable  $\mathfrak{f}_r$ -module of constant Jordan type. Then if  $\mathfrak{sl}(1|1)_{\overline{0}}$  has basis  $\{t_1, \dots, t_r\}$  and  $\mathfrak{sl}(1|1)_{\overline{1}}$  has basis  $\{x_1, \dots, x_r, y_1, \dots, y_r\}$  such that  $[x_i, y_i] = t_i$ , then  $t_i.m = 0$  for all  $m \in M$ .*

*Proof.* Since  $M$  has constant Jordan type with  $a_1 > 0$ , then let  $v$  be a generator for one of the trivial summands in the decomposition  $M|_{\langle x_i \rangle} \cong k^{\oplus a_1} \oplus P^{\oplus a_2}$ , so that  $x_i.v = 0$ . Then consider  $x.y.v = y.x.v + [x, y].v = t_i.v$ . If  $y.v = 0$ , then we have  $0 = t_i.v$  and if  $w = y.v \neq 0$  then  $x.w = t_i.v$ , but since  $v$  is a trivial summand as an  $\langle x_i \rangle$ -modules, then it cannot be in the image of a nonzero vector and thus  $t_i.v = 0$  again. Since this is true for all  $1 \leq i \leq r$ , and  $M$  is indecomposable, we conclude that  $t_i.m = 0$  for all  $1 \leq i \leq r$  and  $m \in M$ .  $\square$

We now make the same reduction used in 3.4 since we now know any module of constant Jordan type is in the principal block. As before, let  $V(\mathfrak{a}_s)$  denote an exterior algebra with  $s$  elements generated by  $\langle 1, x_1, \dots, x_s \rangle$  as an algebra and we consider representations of  $U(\mathfrak{f}_r)$  in the principal block as representations of  $V(\mathfrak{a}_{2r})$ . Furthermore, recall that each of the  $a_i$  are odd and thus act on a supermodule  $M$  via an odd endomorphism.

Because of this general reduction, we can say more about  $\mathfrak{f}$ -modules of constant Jordan type if we require that  $M|_{\langle x \rangle} \cong k^{\oplus a_1} \oplus P^{\oplus a_2}$  for all  $x \in \mathfrak{f}_{\overline{1}}$  which is the self commuting cone  $\mathcal{X}$  associated to  $V(\mathfrak{a})$ .

We refer to  $U(\mathfrak{f})$ -modules which satisfy  $M|_{\langle x \rangle} \cong k^{\oplus a_1} \oplus P^{\oplus a_2}$  for all  $x \in \mathfrak{f}_{\overline{1}}$  as  $\mathfrak{f}$ -modules of *strong constant Jordan type* when needed. For simplicity and full generality, much of the following is treated in the context of  $V(\mathfrak{a})$ -supermodules and for such modules, the term constant Jordan type automatically implies the stronger condition under the correspondence between  $\mathfrak{f}$  and  $\mathfrak{a}$ .

**Proposition 5.2** (Benson). *Let  $M$  and  $N$  be  $V(\mathfrak{a})$ -modules. Then  $M$  and  $N$  both have constant Jordan type if and only if  $M \oplus N$  has constant Jordan type.*

*Proof.* The proof is the same as in [2, Theorem 4.1.9] as the result is based on [2, Theorem 3.6.3] which holds for exterior algebras.  $\square$

*Remark.* This generalization eliminates the counter examples presented in 3.3. This can be seen by considering each of the modules presented in Examples 3.10, 3.11, and 3.12 restricted to  $x + y$ . In these cases, the Jordan type at  $x + y$  is  $2[2]$ ,  $2[1] + 2[2]$ , and  $4[2]$  respectively and so none of the examples are of constant Jordan type when  $\mathcal{X}$  is enlarged to all of  $\mathfrak{f}_{\overline{\mathbb{T}}}$ .

**5.2. Super Vector Bundles.** An important application of modules of constant Jordan type is their use in construction vector bundles over projective space  $\mathbb{P}^n$  given by a construction first introduced in [21] and detailed in [2, Section 7]. In keeping with the convention in [2], we take the term super vector bundle to mean locally free sheaf of finite super rank  $(m|n)$ , i.e. a sheaf of  $\mathcal{O}$ -supermodules which is locally free and has  $m$  even basis elements and  $n$  odd basis elements.

The setting of  $V(\mathfrak{a})$ -modules of constant Jordan type provides two different analogous construction which are considered here. The first possibility is to use such modules to find super vector bundles over a super manifold, which quickly fails. The other is accomplished by adapting the construction in [2, Section 7] to fit the context of  $V(\mathfrak{a})$ -modules.

**5.2.1. Super Manifolds.** First we construct a natural super  $k$ -manifold associated to  $V(\mathfrak{a}_s)$  over which the super vector bundles should lie. Note that  $V(\mathfrak{a}_s)$  already is isomorphic to an odd coordinate system of a splitting neighborhood (in the terminology of [28]). Thus, we can view  $V(\mathfrak{a}_s)$  as a sheaf of super commutative  $k$ -algebras over single pointed super  $k$ -manifold. Then the pair

$$\mathcal{M} := (\mathrm{Spec} V(\mathfrak{a}_s), V(\mathfrak{a}_s))$$

is a super  $k$ -manifold of dimension  $(0|s)$ , since  $V(\mathfrak{a}_s)$  is nilpotent and thus has trivial spectrum, or in other words,  $\mathcal{M}$  is a super manifold consisting of a single point and nilpotent “fuzz.”

At this point, the construction has effectively failed since in the super manifold setting, the vector spaces obtained by considering the fibers  $M_x = \mathrm{Ker}(x)/\mathrm{Im}(x)$  for each  $x \in V(\mathfrak{a}_s)$  are now concentrated topologically over one point.

**5.2.2. Super Vector Bundles over Projective Space.** Instead, we now consider extending the construction detailed in [2, Section 7] which is particularly useful for the setting of  $V(\mathfrak{a})$ -modules. First, a slight generalization of the algebraic vector bundles considered in [2] must be defined.

**Definition 5.3.** Let  $X$  be a connected reduced Noetherian scheme with structure sheaf  $\mathcal{O}_X$ . Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_X$  modules which is locally free of rank  $a$ . Then  $\mathcal{F}$  is called a *vector bundle* of rank  $a$ .

Furthermore, if a sheaf  $\mathcal{F}$  of  $\mathbb{Z}_2$  graded modules and is locally free of rank  $(r_{ev}|r_{od})$ , where there are  $r_{ev}$  even basis elements and  $r_{od}$  odd ones, then we say  $\mathcal{F}$  is a *super vector bundle* of rank  $(r_{ev}|r_{od})$ .

Note that since we are working with standard schemes, we will be consider the structure sheaf to be concentrated in the even degree for the purposes of introducing sheaves of  $\mathbb{Z}_2$



graded  $\mathcal{O}_X$ -modules. This means that  $\mathcal{O}_X(U)$  will act by an even endomorphism on  $\mathcal{F}(U)$  for any open  $U \subseteq X$ .

The vector bundles are constructed as follows. Let  $V(\mathfrak{a}_s)$  be as above. Then the Jacobson radical

$$J = J(V(\mathfrak{a}_s)) = \bigoplus_{i>0} \Lambda^i(\mathfrak{a}_s)$$

is generated by  $\langle x_1, \dots, x_s \rangle$  and  $J/J^2$  has a basis of  $\{\bar{x}_1, \dots, \bar{x}_s\}$  where  $\bar{x}_i$  denotes the image of  $x_i$  in  $J/J^2$ . For

$$\alpha = (\lambda_1, \dots, \lambda_s) \in \mathbb{A}^s \setminus \{0\}$$

define

$$x_\alpha = \lambda_1 x_1 + \dots + \lambda_s x_s \in J$$

which satisfies  $x_\alpha^2 = 0$  by construction.

Recall that the cohomology ring for the superalgebra  $V(\mathfrak{a}_s)$ , whose construction and subsequent computation is given in [5, Theorem 2.5.2] as

$$H^\bullet(\mathfrak{a}_s, (\mathfrak{a}_s)_{\bar{0}}; k) \cong S(\mathfrak{a}_s^*)^{(\mathfrak{a}_s)_{\bar{0}}}.$$

There are no invariants since there is no even component of  $\mathfrak{a}_s$  and so the cohomology is given by symmetric functions on the odd generators of the Lie superalgebra  $\mathfrak{a}_s$ , which in this case is the whole superalgebra.

Thus there is an isomorphism  $S(\mathfrak{a}_s^*) \cong k[Y_1, \dots, Y_s]$  where the  $Y_i$  are linear functions defined by  $Y_i(x_j) = \delta_{ij}$ . We also observe that this can be thought of as the coordinate ring of an affine space of dimension  $s$  with basis elements  $x_i$ . We denote this space by  $\mathbb{A}^s$  and let  $\mathbb{P}^{s-1}$  denote the associated projective space with corresponding structure sheaf  $\mathcal{O}$  and twists  $\mathcal{O}(j)$ . For a  $V(\mathfrak{a}_s)$  module  $M$ , let  $\tilde{M}$  denote the super vector bundle  $M \otimes \mathcal{O}$  and  $\tilde{M}(j) = M \otimes \mathcal{O}(j)$  for the  $j$ th twist of the sheaf  $\tilde{M}$ .

Continuing to follow [2], we define  $\theta_M : \tilde{M}(j) \rightarrow \tilde{M}(j+1)$  by

$$\theta_M(m \otimes f) = \sum_{i=1}^s x_i \cdot m \otimes Y_i f$$

for all  $j \in \mathbb{Z}$ . Note that this is an odd morphism of sheaves of supermodules because if  $m$  is homogeneous,  $x_i \cdot m$  has opposite parity from  $m$  for all  $i$  and  $Y_i f$  is necessarily even since  $\mathcal{O}$  is by definition.

As shown in [2, Section 7.3], we can identify the fibers of  $\mathcal{O}(j)$  at  $\bar{\alpha}$  with  $k$  (concentrated in even degree since  $\mathcal{O}$  is even) by fixing a choice of  $\alpha$  lying over  $\bar{\alpha}$ . Additionally, since  $\tilde{M}(j) = M \otimes \mathcal{O}(j)$ , this choice of  $\alpha$  gives an identification of the fiber of  $\tilde{M}(j)$  at  $\bar{\alpha}$  with  $M$  and subsequently, the action of  $\theta_M$  on the fiber at a point  $\bar{\alpha}$  is given by multiplication by  $x_\alpha$ , detailed as follows for the case of  $\mathcal{O}$  (see [2] for the slightly more general case of  $\mathcal{O}(j)$ ).

For a point  $\bar{\alpha} \in \mathbb{P}^{s-1}$ , consider an affine patch containing this point,  $\mathcal{O}(U_\ell)$  where the  $\ell$ th coordinate does not vanish. If we make the identification

$$\mathcal{O}(U_\ell) \cong k[U_\ell] \cong k[Y_1 Y_\ell^{-1}, \dots, \widehat{Y_\ell Y_\ell^{-1}}, \dots, Y_s Y_\ell^{-1}],$$

then the fiber of  $\mathcal{O}$  at  $\bar{\alpha}$  is given by

$$k[U_\ell] \otimes_{k[U_\ell]} k \cong k$$

where  $k$  is a  $k[U_\ell]$ -module by evaluation at  $x_{\bar{\alpha}}$ . Then the fiber of  $M \otimes \mathcal{O}$  at  $\bar{\alpha}$  is isomorphic to

$$M \otimes k[U_\ell] \otimes_{k[U_\ell]} k \cong M \otimes k \cong M$$

and so the action of  $\theta_M$  on this fiber sends  $m \otimes 1 \otimes_{k[U_\ell]} 1$  to

$$\begin{aligned} \sum_i x_i \cdot m \otimes Y_i \otimes_{k[U_\ell]} 1 &= \sum_i x_i \cdot m \otimes 1 \otimes_{k[U_\ell]} \lambda_i \\ &= \sum_i \lambda_i x_i \cdot m \otimes 1 \otimes_{k[U_\ell]} 1 = x_\alpha \cdot m \otimes 1 \otimes_{k[U_\ell]} 1 \end{aligned}$$

for some choice of  $\alpha$  lying over  $\bar{\alpha}$ . Since the image and kernel of multiplication by  $x_\alpha$  on  $M$  are invariant under scaling and hence the choice of  $\alpha$ , we can use this observation to construct some interesting functors. Furthermore, this computation implies that  $\theta_M^2 \equiv 0$  since  $x_\alpha^2 = 0$  for any  $\alpha \in \mathbb{A}^s \setminus \{0\}$ .

Continuing, we define functors  $\mathcal{F}_i$  for  $i = 1$  or  $2$  from  $V(\mathfrak{a}_s)$ -supermodules to coherent sheaves of supermodules on  $\mathbb{P}^{s-1}$  by

$$\mathcal{F}_i(M) = \frac{\text{Ker } \theta_M \cap \text{Im } \theta_M^{i-1}}{\text{Im } \theta_M^i}.$$

Note that for  $i = 1$  this gives the functor

$$\mathcal{F}_1(M) = \frac{\text{Ker } \theta_M}{\text{Im } \theta_M}$$

and when  $i = 2$  this is simply

$$\mathcal{F}_2(M) = \text{Im } \theta_M$$

since  $\text{Im } \theta_M \subseteq \text{Ker } \theta_M$  and  $\theta_M^2 \equiv 0$  as a map of sheaves of supermodules. This is also why we do not define  $\mathcal{F}_i$  for  $i > 2$  because the corresponding generalization gives sheaves which are identically zero and thus of no interest.

For  $\alpha \in \mathbb{A}^s \setminus \{0\}$  with residue field  $k(\bar{\alpha})$  where  $\bar{\alpha}$  is the image of  $\alpha$  in  $\mathbb{P}^{s-1}$ , the corresponding construction for specialization is given by defining maps

$$\begin{aligned} x_\alpha : M \otimes k(\bar{\alpha}) &\rightarrow M \otimes k(\bar{\alpha}) \\ m \otimes v &\mapsto x_\alpha \cdot m \otimes v \end{aligned}$$

for each  $\alpha \in \mathbb{A} \setminus \{0\}$  and a functor from  $V(\mathfrak{a}_s)$ -supermodules to super vector spaces

$$\mathcal{F}_{i,\alpha}(M) = \frac{\text{Ker } x_\alpha \cap \text{Im } x_\alpha^{i-1}}{\text{Im } x_\alpha^i}$$

which only depends on the image  $\bar{\alpha} \in \mathbb{P}^{s-1}$  by construction. Note that  $\mathcal{F}_{1,\alpha}(M) \cong M_{x_\alpha}$  as super vector spaces. These functors are particularly interesting in relation to modules of constant Jordan type in which case the resulting sheaves are actually (super) vector bundles.

The proof of the main theorem (Theorem 5.5) relies on another theorem analogous to that of [2, Theorem 5.2.2] but for the case of super vector bundles over projective space.

**Theorem 5.4.** *Let  $\mathbb{P}^{s-1}$  denote the projectivization of affine  $s$  space  $\mathbb{A}^s$  with structure sheaf  $\mathcal{O}$ .*

- (1) *If  $\mathcal{F}$  is the coherent sheaf of  $\mathcal{O}$ -supermodules of, then the following are equivalent.*
  - (a) *The sheaf  $\mathcal{F}$  is a super vector bundle of rank  $(r_{ev}|r_{od})$ .*
  - (b) *The even and odd dimensions of the fiber  $\dim_{k(\bar{\alpha})} \mathcal{F}_{\bar{\alpha}} \otimes_{\mathcal{O}_{\bar{\alpha}}} k(\bar{\alpha})$  is constant for all  $\bar{\alpha} \in \mathbb{P}^{s-1}$ .*
- (2) *If  $f : \mathcal{F} \rightarrow \mathcal{F}'$  is an odd map of super vector bundles (where  $\mathcal{F}'$  has rank  $(r_{ev}|r_{od})$ ) on  $\mathbb{P}^{s-1}$  then the following are equivalent.*

- (a) The cokernel of  $f$  is a super vector bundle of rank  $(r_{ev} - r'_{od} | r_{od} - r'_{ev})$ .
- (b) The induced map of fibers

$$\bar{f} : \mathcal{F}_{\bar{\alpha}} \otimes_{\mathcal{O}_{\bar{\alpha}}} k(\bar{\alpha}) \rightarrow \mathcal{F}'_{\bar{\alpha}} \otimes_{\mathcal{O}_{\bar{\alpha}}} k(\bar{\alpha})$$

has constant rank  $(r'_{ev} | r'_{od})$  for all  $\bar{\alpha} \in \mathbb{P}^{s-1}$ .

If these hold, then the image of  $f$  is a super vector bundle of rank  $(r'_{ev} | r'_{od})$ .

*Proof.* We proceed similarly to [2]. Define a function  $\phi : \mathbb{P}^{s-1} \rightarrow \mathbb{Z} \times \mathbb{Z}$  by assigning to each point  $\bar{\alpha} \in \mathbb{P}^{s-1}$  the pair

$$(\dim_{k(\bar{\alpha})}(\mathcal{F}_{\bar{\alpha}} \otimes_{\mathcal{O}_{\bar{\alpha}}} k(\bar{\alpha}))_{\bar{0}}, \dim_{k(\bar{\alpha})}(\mathcal{F}_{\bar{\alpha}} \otimes_{\mathcal{O}_{\bar{\alpha}}} k(\bar{\alpha}))_{\bar{1}}).$$

For any coherent sheaf  $\mathcal{F}$  and for any  $m, n \in \mathbb{Z}$ , the sets

$$\mathbb{P}_{\bar{0} < m}^{s-1} = \{\bar{\alpha} \in \mathbb{P}^{s-1} \mid \dim_{k(\bar{\alpha})}(\mathcal{F}_{\bar{\alpha}} \otimes_{\mathcal{O}_{\bar{\alpha}}} k(\bar{\alpha}))_{\bar{0}} < m\}$$

$$\mathbb{P}_{\bar{1} < n}^{s-1} = \{\bar{\alpha} \in \mathbb{P}^{s-1} \mid \dim_{k(\bar{\alpha})}(\mathcal{F}_{\bar{\alpha}} \otimes_{\mathcal{O}_{\bar{\alpha}}} k(\bar{\alpha}))_{\bar{1}} < n\}$$

are open, seen as follows.

It is sufficient to show the claim when over an affine variety  $X = \text{Spec } R$  where  $R$  is Noetherian and  $\mathcal{F} = \tilde{M}$ , the sheaf associated to a finitely generated  $R$ -supermodule  $M$ . Finally, recall that  $R$  acts evenly on  $\mathcal{F}$ , as this key fact will be used repeatedly without further comment.

If  $\bar{\alpha} \in X \cap \mathbb{P}_{\bar{0} < m}^{s-1}$  has corresponding prime ideal  $\mathfrak{p}$  of  $R$ , then  $\mathcal{F}_{\bar{\alpha}} \otimes_{\mathcal{O}_{\bar{\alpha}}} k(\bar{\alpha}) \cong M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ . Let  $m' = \dim_{k(\bar{\alpha})}(M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}})_{\bar{0}}$  so  $m' < m$  by definition. Let  $\tilde{v}_1, \dots, \tilde{v}_{m'}$  and  $\tilde{w}_1, \dots, \tilde{w}_{n'}$  be a homogeneous set of elements (concentrated in even and odd degree, respectively) of  $M_{\mathfrak{p}}$  such that the images  $\bar{v}_1, \dots, \bar{v}_{m'}$  in  $(M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}})_{\bar{0}}$  form a basis for this space. By Nakayama's lemma,  $\tilde{v}_1, \dots, \tilde{v}_{m'}$  generate  $(M_{\mathfrak{p}})_{\bar{0}}$  and since there are a finite number of the  $v_i$ , by clearing the denominators we can assume that these are the images of a set of homogeneous even elements  $v_1, \dots, v_{m'} \in M$ .

If  $y_1, \dots, y_{d_{ev}}, z_1, \dots, z_{d_{od}}$  is a homogeneous basis for  $M$ . The image of each  $y_i$  in  $M_{\mathfrak{p}}$  is a linear combination of the  $\tilde{v}_i$  with coefficients in  $R_{\mathfrak{p}}$ . Again, each of these linear combination has only a finite number of elements so we can clear the denominators by multiplying by a single homogeneous even element  $r \in R$  with  $r \notin \mathfrak{p}$  so that each of the  $ry_i$  are linear combinations of the  $v_i$  with coefficients in  $R$ .

Then define  $U_r$  to be the open set consisting of prime ideals  $\mathfrak{q}$  such that the for the fixed  $r$  above,  $r \notin \mathfrak{q}$ . Then for each  $\mathfrak{q} \in U_r$ , we can clear the denominators as described above and write the images of the  $y_i$  as linear combinations of the  $v_i$  with coefficients in  $R_{\mathfrak{q}}$  which shows that  $M_{\mathfrak{q}}$  is generated at most by  $m'$  even elements and that  $U_r \subseteq X \cap \mathbb{P}_{\bar{0} < m}^{s-1}$ . Thus, we have shown that  $X_i \cap \mathbb{P}_{\bar{0} < m}^{s-1}$  for each  $X_i$  in an open affine covering of  $\mathbb{P}^{s-1}$ , hence  $\mathbb{P}_{\bar{0} < m}^{s-1}$  is open and similarly  $\mathbb{P}_{\bar{1} < n}^{s-1}$  is as well. With this fact established, we proceed to the claims of the theorem.

(1a)  $\Rightarrow$  (1b) Let  $(S)^c$  denote the compliment of a set  $S$ . By assumption, for each  $\bar{\alpha} \in \mathbb{P}^{s-1}$  there is an open neighborhood of  $\bar{\alpha}$  on which  $\mathcal{F}$  is free and thus,  $\phi$  is constant. By composing  $\phi$  with projections onto the separate factors, we get maps  $\phi_{\bar{0}} = \pi_{\bar{0}} \circ \phi$  and  $\phi_{\bar{1}} = \pi_{\bar{1}} \circ \phi$  which are constant as well. Thus, there are open neighborhoods around each point of the sets  $(\mathbb{P}_{\bar{0} < m}^{s-1})^c$  and  $(\mathbb{P}_{\bar{1} < n}^{s-1})^c$  for each  $m, n \in \mathbb{Z}$  such that these sets are nonempty. Therefore these sets are open as well and since  $\mathbb{P}^{s-1}$  is connected, each of these nonempty sets must be all of  $\mathbb{P}^{s-1}$  and so  $\phi_{\bar{0}}$  and  $\phi_{\bar{1}}$  are constant which yields that  $\phi$  is constant as well.

(1b)  $\Rightarrow$  (1a) Now we assume that  $\phi$  as defined above is constant on  $\mathbb{P}^{s-1}$ . Again it suffices to show that  $\mathcal{F}$  is locally free of rank  $\phi(\bar{\alpha})$  on an open affine set since this is a local condition. Thus let  $X = \text{Spec } R$  for some Noetherian  $R$  and we assume that  $\mathcal{F} = \tilde{M}$  for some finitely generated supermodule  $M$  where a point  $\bar{\alpha}$  corresponds to  $\mathfrak{p}$ . Let  $v_i, w_j, \tilde{v}_i, \tilde{w}_j$  and  $y_i, z_j$  be as above and so again by clearing denominators, there is some  $r \in R$  such that  $r \notin \mathfrak{p}$  so that the images of  $ry_i$  and  $rz_j$  in  $M_{\mathfrak{p}}$  are linear combinations of the  $\tilde{v}_i$  and  $\tilde{w}_j$ . If  $R_r$  and  $M_r$  denote  $R$  and  $M$  with  $r$  inverted, then the images of the  $x_i$  and  $y_j$  in  $M_r$  are linear combinations of the images of the  $v_i$  and  $w_j$  respectively and thus  $M_r$  is generated by the images of  $v_i$  and  $w_j$ . Since the  $v_i$  are indexed from  $1, \dots, \phi_{\bar{0}}$  and the  $w_i$  from  $1, \dots, \phi_{\bar{1}}$  we obtain a surjective homomorphism and corresponding exact sequence

$$0 \longrightarrow K \longrightarrow R_r^{\phi_{\bar{0}}(\bar{\alpha}) + \phi_{\bar{1}}(\bar{\alpha})} \longrightarrow M_r \longrightarrow 0$$

where the kernel is denoted by  $K$ . We can apply the (exact) functor of localization to obtain the sequence

$$0 \longrightarrow K_{\mathfrak{q}} \longrightarrow R_{\mathfrak{q}}^{\phi_{\bar{0}}(\bar{\alpha}) + \phi_{\bar{1}}(\bar{\alpha})} \longrightarrow M_{\mathfrak{q}} \longrightarrow 0$$

for each prime ideal  $\mathfrak{r} \notin \mathfrak{q}$ . We assumed  $\phi$  was constant, so  $(M_{\mathfrak{q}}/\mathfrak{q}M_{\mathfrak{q}})_{\bar{0}}$  has dimension  $\phi_{\bar{0}}(\bar{\alpha})$  and  $(M_{\mathfrak{q}}/\mathfrak{q}M_{\mathfrak{q}})_{\bar{1}}$  has dimension  $\phi_{\bar{1}}(\bar{\alpha})$  over  $R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}$  and so  $K_{\mathfrak{q}} \subseteq \mathfrak{q}R_{\mathfrak{q}}^{\phi_{\bar{0}}(\bar{\alpha}) + \phi_{\bar{1}}(\bar{\alpha})}$ . Thus, the coordinates of an element in  $K$  are in  $\mathfrak{q}$ . Since  $X \subseteq \mathbb{P}$  is reduced,  $R_r$  has no nilpotent elements, so the intersection of the prime ideals of  $R_r$  is zero and then so is  $K$ . Therefore,  $\mathcal{F}$  is free on the open subset of  $X$  defined by  $r$ .

The proof of part (2) is the same as in [2] with attention given to the even and odd components of the vector bundles.  $\square$

**Theorem 5.5.** *Let  $M$  be a  $V(\mathfrak{a}_s)$ -module of constant Jordan type  $(a_{ev}|a_{od})[1] + a_2[2]$ . Then*

- (1)  $\mathcal{F}_1(M)$  is a super vector bundle of rank  $(a_{od}|a_{ev})$  over  $\mathbb{P}^{s-1}$  with fiber over  $\bar{\alpha}$  isomorphic to  $M_{x_{\alpha}}$ ;
- (2)  $\mathcal{F}_2(M)$  is a vector bundle of rank  $a_2$  over  $\mathbb{P}^{s-1}$  with fiber over  $\bar{\alpha}$  isomorphic to  $\text{Soc}(P|_{\langle x_{\alpha} \rangle})$  where  $M|_{\langle x_{\alpha} \rangle} \cong k^{\oplus a_1} \oplus P^{\oplus a_2}$ ;

Furthermore, if  $f : M \rightarrow N$  is a homogeneous map of (super) modules of constant Jordan type, then for any  $\bar{\alpha} \in \mathbb{P}^{s-1}$  with residue field  $k(\bar{\alpha})$  there is a diagram

$$\begin{array}{ccc} \mathcal{F}_i(M) \otimes_{\mathcal{O}} k(\bar{\alpha}) & \xrightarrow{\mathcal{F}_i(f)} & \mathcal{F}_i(N) \otimes_{\mathcal{O}} k(\bar{\alpha}) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{F}_{i,\alpha}(M) & \longrightarrow & \mathcal{F}_{i,\alpha}(N) \end{array}$$

which commutes.

*Proof.* For (1), by definition

$$M|_{\langle x_{\alpha} \rangle} \cong k_{ev}^{\oplus a_{ev}} \oplus k_{od}^{\oplus a_{od}} \oplus P^{\oplus a_2}$$

for all  $0 \neq \alpha \in \mathbb{A}^s$ . Recall that the fibers of  $M \otimes \mathcal{O}$  are isomorphic  $M$  and that the action of  $\theta_M$  on the fibers is multiplication by  $x_{\alpha}$ , so  $\dim_{k(\bar{\alpha})}(\text{Im } \theta_M)_{\bar{\alpha}} \otimes k(\bar{\alpha}) = a_2$  for all  $\bar{\alpha} \in \mathbb{P}^{s-1}$ . By Theorem 5.4 (1),  $\mathcal{F}_2(M) = \text{Im } \theta_M$  is a vector bundle of rank  $a_2$ .

Consider the short exact sequence

$$0 \longrightarrow \operatorname{Im} \theta_M \longrightarrow \operatorname{Ker} \theta_M \longrightarrow \mathcal{F}_1(M) \longrightarrow 0$$

which defines the functor  $\mathcal{F}_1(M)$ . Applying the right exact functor of specialization to the fiber over  $\bar{\alpha}$  yields a diagram

$$(5.6) \quad \begin{array}{ccccccc} \operatorname{Im} \theta_M \otimes_{\mathcal{O}} k(\bar{\alpha}) & \longrightarrow & \operatorname{Ker} \theta_M \otimes_{\mathcal{O}} k(\bar{\alpha}) & \longrightarrow & \mathcal{F}_1(M) \otimes_{\mathcal{O}} k(\bar{\alpha}) & \longrightarrow & 0 \\ \parallel & & \parallel & & \parallel & & \\ \operatorname{Im} x_{\alpha} & \xrightarrow{\iota} & \operatorname{Ker} x_{\alpha} & \longrightarrow & \mathcal{F}_{1,\alpha}(M) & \longrightarrow & 0 \end{array}$$

where  $\iota$  is injective for all  $\bar{\alpha}$ . Thus,  $\mathcal{F}_{1,\alpha}(M)$  has constant rank  $(a_{ev}|a_{od})$  and so by Theorem 5.4 (2),  $\mathcal{F}_1(M)$  is a super vector bundle of rank  $(a_{ev}|a_{od})$ .

Part (2) follows from observing that

$$\begin{array}{ccc} \tilde{M}(j) & \xrightarrow{\theta_M} & \tilde{M}(j+1) \\ f \otimes \operatorname{Id}_{\mathcal{O}} \downarrow & & \downarrow (-1)^{|f|} f \otimes \operatorname{Id}_{\mathcal{O}} \\ \tilde{N}(j) & \xrightarrow{\theta_N} & \tilde{N}(j+1) \end{array}$$

is a commutative diagram which induces maps  $\operatorname{Im} \theta_M \rightarrow \operatorname{Im} \theta_N$  and  $\operatorname{Ker} \theta_M \rightarrow \operatorname{Ker} \theta_N$  and hence on the cokernels. Then applying the specialization diagram in Equation 5.6 completes the proof.  $\square$

**5.3. Constant Jordan Type for  $\mathfrak{f}_1$ .** Another interesting property of  $\mathfrak{f}_r$ -modules of constant Jordan type is that when  $r = 1$ , this Lie superalgebra is too small to accommodate indecomposable modules of constant Jordan type where  $a_1 > 1$ . One important observation in showing this fact is that if  $M$  is indecomposable and non-projective, then  $\operatorname{Rad}^2(M) = 0$  and furthermore, the parity of  $M$  distinguishes the head and the socle of  $M$ , as seen in the following lemma. Another key fact is that for such modules, the super commutative action of  $\mathfrak{f}_1$  on  $M$  becomes a commutative one since  $xy.m = yx.m = 0$ .

**Lemma 5.7.** *Let  $M$  be an indecomposable non-projective supermodule over  $V(\mathfrak{a}_2) = \langle 1, x, y \rangle$  such that  $\dim(M) > 1$ . Then*

- (1)  $\operatorname{Rad}(M) = \operatorname{Soc}(M)$ ;
- (2) *either  $M_{\bar{0}} = \operatorname{Hd}(M)$  and  $M_{\bar{1}} = \operatorname{Soc}(M)$  or  $M_{\bar{0}} = \operatorname{Soc}(M)$  and  $M_{\bar{1}} = \operatorname{Hd}(M)$ .*

*Proof.* For (1), let  $m \in \operatorname{Rad}(M)$ . Then  $m = (ax + by).n$  for some  $n \in M$  and  $(0, 0) \neq (a, b) \in k^2$ . Then  $x.m = bxy.n$  and  $y.m = ayx.n$  both of which must be zero, otherwise this would be the socle of a projective  $V(\mathfrak{a}_2)$ -module which would split off as a direct summand implying  $M$  is either projective or decomposable. Similarly, if  $m' \in \operatorname{Soc}(M)$  then  $x.m'$  and  $y.m'$  are both zero and since  $M$  is indecomposable,  $m'$  is the image of some  $n' \in M$  under the action of  $V(\mathfrak{a}_2)$  which proves (1).

We show (2) by induction on  $\dim(M)$ . The base case is when  $\dim(M) = 2$ , which is trivial since  $M$  is indecomposable. Let  $\dim(M) > 2$  and let  $m \in \operatorname{Hd}(M)$  and let  $N$  denote the maximal submodule of  $M$  complimentary to  $m$ . Then  $\dim(N) < \dim(M)$  and so by the inductive hypothesis,  $N$  satisfies (2). Since  $M$  is indecomposable,  $(ax + by).m \neq 0$  for some  $(0, 0) \neq (a, b) \in k^2$ . Then  $(ax + by).m \in \operatorname{Soc}(M) \cap N$  and since the action of  $ax + by$  is odd,

then  $m$  has the opposite parity as  $\text{Soc}(N)$  and thus has the same parity as  $\text{Hd}(N)$  which implies that (2) holds for  $M$  as well.  $\square$

**Proposition 5.8.** *Let  $M$  be an indecomposable non-projective supermodule over  $\mathfrak{f} = \mathfrak{sl}(1|1)$  of strong constant Jordan type  $(a_{ev}|a_{od})[1] + a_2[2]$ . Then exactly one of  $a_{ev}$  and  $a_{od}$  is 0, i.e. there cannot be both even and odd trivial summands in the Jordan decomposition of  $M$ .*

*Proof.* First note that since  $M$  is not projective, then  $a_{ev} + a_{od} = a_1 \neq 0$  by Proposition 3.5. Furthermore, if  $a_2 = 0$  then  $a_1 = 1$  since  $M$  is indecomposable and so these cases follow immediately. In order to utilize Lemma 5.7, we consider the strong Jordan decomposition of  $M$ ,

$$M|_{\langle ax+by \rangle} \cong k_{ev}^{\oplus a_{ev}} \oplus k_{od}^{\oplus a_{od}} \oplus P^{\oplus a_2},$$

for all  $0 \neq ax + by \in \mathfrak{f}_{\overline{1}}$ , and we have reduced to considering the situation when both  $a_{ev} + a_{od} \geq 1$  and  $a_2 \geq 1$ . In the remainder of the proof, by dualizing and then applying the parity change functor (if either are necessary), we assume without loss of generality that

$$(1) \dim \text{Hd}(M) \leq \dim \text{Soc}(M); \quad (2) M_{\overline{0}} = \text{Hd}(M) \text{ and } M_{\overline{1}} = \text{Soc}(M).$$

This further implies that

$$(3) \dim \text{Hd}(M) = a_{ev} + a_2; \quad (4) \dim \text{Soc}(M) = a_{od} + a_2; \quad (5) a_{ev} \leq a_{od}.$$

As noted above, the action of  $\mathfrak{f}$  on  $M$  becomes commutative since  $\text{Rad}^2(M) = 0$  so we can apply the theory of generic kernels and images introduced in [9] and summarized in [2, Chapter 4]. Additionally, in this situation since  $M$  is of strong constant Jordan type it is consistent with the notion of constant Jordan type in [2]. Let  $\mathfrak{K}(M)$  denote the generic kernel of  $M$ .

Using [2, Lemma 4.10.12], we can determine exactly  $\mathfrak{K}(M)$ . Define

$$n := \text{the number of Jordan blocks of each } 0 \neq ax + by \in \mathfrak{f} \text{ acting on } M$$

$$d := \dim \mathfrak{K}(M) / \text{Rad}(\mathfrak{K}(M))$$

and the Lemma yields that  $n = d$ . Note that our decomposition gives us that  $n = a_{ev} + a_{od} + a_2$ .

Applying [2, Proposition 4.7.8] to  $M$  yields  $\text{Soc}(M) \subseteq \mathfrak{K}(M)$ . By [2, Theorem 4.7.4],  $\mathfrak{K}(M)$  has the constant image property and [9, Proposition 5.1] gives a classification of such modules. Again, the result from [9] applies here as it is based on [24, Proposition 5] which applies to the general setting we consider here. By the classification,

$$\mathfrak{K}(M) \cong W_{n_1,2} \oplus \dots \oplus W_{n_t,2} \oplus k_{od}^s$$

for some integers  $s, t, n_i$ , where the  $W_{n_i,2}$  are defined in [2, Section 4.11]. Note that  $s$  cannot be 0 or else condition (1) above is violated. Additionally, the image of  $\mathfrak{K}(M)$  under the action of  $\mathfrak{f}$  is exactly  $\oplus_i \text{Ker}(W_{n_i,2})$ , so if  $t \geq 1$ , then since  $\text{Hd}(W_{n_i,2})$  is not in the image of any element of  $M$ , then this produces a direct sum decomposition of  $M$ , a contradiction. Thus,  $\mathfrak{K}(M) \cong k_{od}^s$  and since  $\text{Soc}(M) \subseteq \mathfrak{K}(M)$ , we conclude that in fact  $\text{Soc}(M) = \mathfrak{K}(M)$ . Then  $d_1 = a_{od} + a_2$  by (4) and so  $a_{od} + a_2 = a_{ev} + a_{od} + a_2$  and therefore  $a_{ev} = 0$ . Recalling that  $a_{ev} + a_{od} \geq 1$  and that the reductions made may have changed the parity, we conclude that exactly one of  $a_{ev}$  and  $a_{od}$  is 0.  $\square$

Combining the results in [2, Chapter 4] with Proposition 5.8, the following theorem completely classifying modules of constant Jordan type over  $\mathfrak{f}$  follows quickly.

**Theorem 5.9.** *Let  $M$  be an indecomposable non-projective supermodule over  $\mathfrak{f} = \mathfrak{sl}(1|1)$  of strong constant Jordan type  $a_1[1] + a_2[2]$ . Then  $a_1 = 1$  and hence  $M$  is endotrivial.*

*Proof.* Begin by making the same assumptions (1) - (5) as in the previous proof and thus  $a_{ev} = 0$ . Furthermore, assume that  $M$  is not  $k_{od}$  as this case is trivial.

Let  $\mathfrak{I}(M)$  denote the generic image of  $M$  as defined in [2, Section 4.10]. According to the results from the same section,  $\mathfrak{I}(M) \subseteq \mathfrak{K}(M) = \text{Soc}(M)$ ,  $M/\mathfrak{I}(M)$  has (ungraded) constant Jordan type, and for all  $0 \neq (a, b) \in k^2$ ,  $\mathfrak{I}(M) \subseteq \text{Im}(ax + by)$ .

Assume that  $\mathfrak{I}(M) \neq 0$ . Then the fact that  $\mathfrak{I}(M) \subseteq \text{Im}(ax + by)$  for all  $0 \neq (a, b) \in k^2$  implies that  $\mathfrak{I}(M) \subset \text{Soc}(P^{\oplus a_2})$  in the Jordan decomposition of  $M$  relative to  $ax + by$ . Thus,  $M/\mathfrak{I}(M)$  is of constant Jordan type as a supermodule, and has type

$$(\dim(\mathfrak{I}(M)|_{a_{od}})[1] + (a_2 - \dim(\mathfrak{I}(M)))[2]$$

which is a contradiction, so  $\mathfrak{I}(M) = 0$ .

By dualizing, [2, Lemma 4.10.3] implies that

$$M^* \cong \mathfrak{K}(M^*) \cong W_{n_1,2} \oplus \dots \oplus W_{n_t,2} \oplus k_{od}^s$$

and since  $M$  and therefore  $M^*$  are indecomposable,  $M \cong W_{n,2}^*$  for some  $n \geq 0$ .

Recalling the reductions we made,  $M$  may be isomorphic to  $W_{n,2}$  or  $W_{m,2}^*$  which both have constant Jordan type  $1[1] + (n-1)[2]$ , and the trivial summand may be concentrated in either degree.  $\square$

Note that there are infinitely many nonisomorphic indecomposable endotrivial  $\mathfrak{f}_1$ -modules as proved in [31] (isomorphic to the  $W_{n,2}$  and  $W_{n,2}^*$ ) and that for  $\mathfrak{f}_r$  when  $r > 1$ , there are indeed indecomposable modules of constant Jordan type  $(a_{ev}|_{a_{od}})[1] + a_2[2]$  where  $a_{ev}, a_{od} \geq 1$  as constructed in Section 3.4.

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